# Governing Equation of 2D Kelvin-Helmholtz instability in compressible plasmas

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## 1 Derivation of the equation

There is an very good paper (Miura and Pritchett, 1982) which gives the 2D ODE that describes KHI with general 2D configuration ($\rho_0(y)$, $p_0(y)$, $\mathbf{u_0} = u_0(y)\hat{e}_x$ and $\mathbf{B_0} = B_{0x}(y)\hat{e}_x + B_{0z}(y)\hat{e}_z$). In there model they also allow inclined modes, i.e., $k_z \neq 0$. The ODE they derived is for the total pressure $p_1^T$. Here we adopt a simpler configuration: $\rho_0(y)$, $p_0(y)$, $\mathbf{u_0} = u_0(y)\hat{e}_x$ and $\mathbf{B_0} = B_0(y)\hat{e}_x$ and consider only the parallel modes ($\mathbf{k} = k\hat{e}_x$). The 0th order fields are time-independent and satisfy:

$$p_0 + \frac{1}{2}B_0^2 = Const \tag{1}$$

i.e.,

$$p_0' + B_0 B_0' = 0 \tag{2}$$

Fourier transform the 1st order quantities such that

$$f_1 = \int_{k,\omega} \hat{f}_1(y,k,\omega)\exp(i(kx-\omega t)) \tag{3}$$

For convenience, define the Doppler shifted frequency:

$$\Omega(y) = \omega - k u_0(y) \tag{4}$$

and the phase velocity $v_p(y) = \Omega/k = \omega/k - u_0(y)$ such that

$$\Omega' = -k u_0', \quad v_p' = -u_0' \tag{5}$$

Then the linearized MHD equation can be written as (neglect all the hats)

$$-i\Omega\rho_1 + ik\rho_0 u_1 + (\rho_0 v_1)' = 0 \tag{6a}$$

$$-i\Omega p_1 + p_0' v_1 + \kappa p_0(iku_1 + v_1') = 0 \tag{6b}$$

$$-i\Omega u_1 + u_0' v_1 + \frac{1}{\rho_0}(ikp_1 - B_0' b_y) = 0 \tag{6c}$$

$$-i\Omega\rho_0 v_1 + p_1' + (B_0 b_x)' - ikB_0 b_y = 0 \tag{6d}$$

$$-i\Omega b_x + B_0' v_1 + B_0 v_1' - u_0' b_y = 0 \tag{6e}$$

$$-i\Omega b_y - ikB_0 v_1 = 0 \tag{6f}$$

We can replace the equation of $b_x$ by the divergence-free condition:

$$ikb_x + b_y' = 0 \tag{7}$$

Our goal is to get a *2nd order ODE for* $v_1$. From Eq (6f), we get:

$$b_y = -\frac{B_0}{v_p} v_1 \tag{8}$$

From Eq (7):

$$b_x = \frac{i}{k} b_y' = -i(\frac{B_0}{\Omega} v_1)' \tag{9}$$

From Eq (6c):

$$u_1 = -\frac{i}{\Omega}[u_0' v_1 + \frac{1}{\rho_0}(ikp_1 - B_0' b_y)] \tag{10}$$

From Eq (6b):

$$p_1 = -\frac{i}{\Omega}(p_0' v_1 + \kappa p_0 v_1') + \frac{\kappa p_0}{v_p} u_1 \tag{11}$$

Use Eq (10) to replace $u_1$ in the above equation, we get:

$$(1 - \frac{c_s^2}{v_p^2})p_1 = -\frac{i}{\Omega}[p_0'(1 - \frac{c_s^2}{v_p^2})v_1 + \kappa p_0 v_1' + \frac{\kappa p_0}{v_p} u_0' v_1] \tag{12}$$

or

$$p_1 = -\frac{i}{\Omega}[p_0' v_1 + \rho_0 \frac{c_s^2}{1 - \frac{c_s^2}{v_p^2}}(v_1' + \frac{u_0'}{v_p} v_1)] \tag{13}$$

where we have used Eq (2). The final step is the replace $p_1$, $b_x$ and $b_y$ in Eq (6d) by Eq (13, 9 and 8). First,

$$(B_0 b_x)' - ikB_0 b_y = -i[B_0'(\frac{B_0}{\Omega}v_1)' + B_0(\frac{B_0}{\Omega}v_1)'' - k^2 \frac{B_0^2}{\Omega} v_1] \tag{14}$$

where

$$(\frac{B_0}{\Omega}v_1)' = \frac{1}{\Omega}[(B_0' + \frac{B_0 u_0'}{v_p})v_1 + B_0 v_1'] \tag{15a}$$

$$(\frac{B_0}{\Omega}v_1)'' = \frac{1}{\Omega}\left\{[2\frac{B_0' u_0'}{v_p} + 2B_0(\frac{u_0'}{v_p})^2 + \frac{B_0 u_0''}{v_p} + B_0'']v_1 + 2(B_0' + \frac{B_0 u_0'}{v_p})v_1' + B_0 v_1''\right\} \tag{15b}$$

So we get

$$(B_0 b_x)' - ikB_0 b_y = -\frac{i}{\Omega}\left\{B_0^2 v_1'' + (3B_0 B_0' + 2B_0^2 \frac{u_0'}{v_p})v_1' + H(y)v_1\right\} \tag{16}$$

where

$$H(y) = (B_0')^2 + B_0 B_0'' + B_0^2 \frac{u_0''}{v_p} + 3\frac{B_0 B_0'}{v_p}u_0' + 2B_0^2(\frac{u_0'}{v_p})^2 - k^2 B_0^2 \tag{17}$$

Then we calculate $p_1'$. For convenience, define two functions:

$$G(y) = \frac{c_s^2}{1 - \frac{c_s^2}{v_p^2}} \tag{18}$$

and

$$h(y) = p_0' + \frac{u_0'}{v_p}\rho_0 G(y) \tag{19}$$

After some calculations, we get:

$$p_1' = -\frac{i}{\Omega}\left\{(h' + \frac{u_0'}{v_p}h)v_1 + [2h + (\rho_0 G - p_0)']v_1' + \rho_0 G v_1''\right\} \tag{20}$$

Finally, insert Eq (16 and 20) in Eq (6d) and clean up, we get:

$$(G + V_A^2)v_1'' + \frac{1}{\rho_0}\big[2h + (\rho_0 G - p_0)' + B_0(3B_0' + 2B_0\frac{u_0'}{v_p})\big]v_1' + \big[\Omega^2 + \frac{1}{\rho_0}H(y) + \frac{1}{\rho_0}(h' + \frac{u_0'}{v_p}h)\big]v_1 = 0 \tag{21}$$

where

$$V_A^2 = \frac{B_0^2}{\rho_0} \tag{22}$$

is the square of the Alfvén speed. For reference,

$$G'(y) = 2G^2(\frac{c_s'}{c_s^3} - \frac{v_p'}{v_p^3}) \tag{23}$$

and

$$h' = p_0'' + \rho_0 G\big[\frac{u_0''}{v_p} + (\frac{u_0'}{v_p})^2\big] + \frac{u_0'}{v_p}(\rho_0' G + \rho_0 G') \tag{24}$$

In analyzing KHI, we need to know the asymptotic behavior of the solution in order to define the boundary conditions. Far from the shear layer, all the background quantities are uniform so Eq (21) reduce to:

$$(G + V_A^2)v_1'' + (\Omega^2 - k^2 V_A^2)v_1 = 0 \tag{25}$$

whose solution is:

$$v_1(y) = \bar{v}_1 \exp\left( \pm \sqrt{\frac{k^2 V_A^2 - \Omega^2}{G + V_A^2}} y \right) \tag{26}$$

and we need to take the decaying branch.