Material Derivatives of line, surface and volume elements in a fluid

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January 25, 2018

We are considering the material derivatives of the line, surface and volume elements inside a flow field $\mathbf{u}(\mathbf{r}, t)$.

1 Line

Consider a line element:

$$\mathbf{\Delta l} = \mathbf{r_2} - \mathbf{r_1} \tag{1}$$

The material derivative is then

$$\begin{aligned} \frac{d\mathbf{\Delta l}}{dt} &= \frac{d\mathbf{r_2}}{dt} - \frac{d\mathbf{r_1}}{dt} \\ &= \mathbf{u_2} - \mathbf{u_1} \\ &= \mathbf{\Delta l} \cdot \nabla \mathbf{u} \end{aligned} \tag{2}$$

2 Volume

The change of a volume ΔV is due to the expansion effect of the flow. Thus when we do the derivative of Δx, Δy, Δz we should consider the longitudinal(compressional) direction only. Write

$$\Delta V = \Delta x \Delta y \Delta z \tag{3}$$

we get

$$\begin{aligned} \frac{d\Delta V}{dt} &= \Delta y \Delta z \left(\frac{d\Delta x}{dt}\right)_{compressional} + \cdots \\ &= \Delta y \Delta z \Delta x \frac{\partial u_x}{\partial x} + \cdots \\ &= \Delta y \Delta z \Delta x \left(\frac{\partial u_x}{\partial x} + \frac{\partial u_y}{\partial y} + \frac{\partial u_z}{\partial z}\right) \\ &= (\nabla \cdot \mathbf{u}) \Delta V \end{aligned} \tag{4}$$

which means that the change of a volume is due to the divergence of the flow.

3 Surface

The material derivative of a surface is the hardest one to deal with because we need to consider both the change of the area and the change of the surface normal direction at the same time. Write

$$\mathbf{\Delta S} = \Delta S \mathbf{n} \tag{5}$$

We then get

$$\frac{d\mathbf{\Delta S}}{dt} = \mathbf{n} \frac{d\Delta S}{dt} + \Delta S \frac{d\mathbf{n}}{dt} \tag{6}$$

3.1 The rate of change of the area ΔS

Consider one point at boundary of the surface volume where the line element along boundary is $\mathbf{\Delta l}$. The change of the area is caused by the flow velocity perpendicular to $\mathbf{\Delta l}$ and along the plane of the surface, i.e., the velocity along the direction of

$\mathbf{\Delta l} \times \mathbf{n}$. We then get:

$$
\begin{aligned}
\frac{d\Delta S}{dt} &= \oint \mathbf{u} \cdot (\mathbf{\Delta l} \times \mathbf{n}) \\
&= \oint \mathbf{\Delta l} \cdot (\mathbf{n} \times \mathbf{u}) \\
&= \int d\mathbf{S} \cdot \nabla \times (\mathbf{n} \times \mathbf{u}) \\
&= \int d\mathbf{S} \cdot [\mathbf{n}(\nabla \cdot \mathbf{u}) - \mathbf{n} \cdot \nabla \mathbf{u}] \\
&= \mathbf{\Delta S} \cdot \mathbf{n}(\nabla \cdot \mathbf{u}) - \mathbf{n} \cdot \nabla \mathbf{u} \cdot \mathbf{\Delta S} \\
&= \Delta S(\nabla \cdot \mathbf{u}) - \mathbf{n} \cdot \nabla \mathbf{u} \cdot \mathbf{\Delta S}
\end{aligned} \tag{7}
$$

3.2 The rate of change of the normal direction n

For simplicity, assume a surface element whose boundary is a perfect circle. Establish a cylindrical coordinate system such that $\mathbf{n}$ is along $\hat{e}_z$ and the center of the surface is at O. The change of $\mathbf{n}$ is due to the velocity along $\mathbf{n}$ direction:

$$
(\frac{d\mathbf{n}}{dt})_{\mathbf{r_0}+\Delta r\hat{e}_r} = \alpha \frac{\mathbf{n} \cdot \mathbf{u}(\mathbf{r}_0 + \Delta r \hat{e}_r)}{\Delta r}(-\hat{e}_r) \tag{8}
$$

(the factor α represents an averaging process over the surface). To get $d\mathbf{n}/dt$ we must integrate in θ direction from 0 to 2π:

$$
\begin{aligned}
\frac{d\mathbf{n}}{dt} &= \alpha \oint d\theta \frac{\mathbf{n} \cdot \mathbf{u}(\mathbf{r}_0 + \Delta r \hat{e}_r)}{\Delta r}(-\hat{e}_r) \\
&= \alpha \oint d\theta \frac{\mathbf{n} \cdot [\mathbf{u}(\mathbf{r}_0) + \Delta r \hat{e}_r \cdot \nabla \mathbf{u}]}{\Delta r}(-\hat{e}_r) \\
&= \alpha \oint d\theta \frac{\mathbf{n} \cdot \mathbf{u}(\mathbf{r}_0)}{\Delta r}(-\hat{e}_r) + \oint d\theta \frac{\Delta r \hat{e}_r \cdot \nabla \mathbf{u} \cdot \mathbf{n}}{\Delta r}(-\hat{e}_r) \\
&= -\alpha \oint d\theta \hat{e}_r(\hat{e}_r \cdot \nabla \mathbf{u} \cdot \mathbf{n})
\end{aligned} \tag{9}
$$

(note that $\oint \hat{e}_r = 0$). Now let's look at $\hat{e}_r \cdot \nabla \mathbf{u} \cdot \mathbf{n}$ and keep in mind that $\mathbf{n} = \hat{e}_z$:

$$
\begin{aligned}
(\hat{e}_r \cdot \nabla \mathbf{u}) \cdot \mathbf{n} &= (\frac{\partial u_r}{\partial r}\hat{e}_r + \frac{\partial u_\theta}{\partial r}\hat{e}_\theta + \frac{\partial u_z}{\partial r}\hat{e}_z) \cdot \mathbf{n} \\
&= \frac{\partial u_z}{\partial r}
\end{aligned} \tag{10}
$$

Then, notice that in the local Cartesian coordinates

$$
\hat{e}_r = \hat{e}_x \cos\theta + \hat{e}_y \sin\theta \tag{11}
$$

and

$$
\begin{aligned}
\frac{\partial}{\partial r} &= \frac{\partial}{\partial x}\frac{\partial x}{\partial r} + \frac{\partial}{\partial y}\frac{\partial y}{\partial r} \\
&= \cos\theta \frac{\partial}{\partial x} + \sin\theta \frac{\partial}{\partial y}
\end{aligned} \tag{12}
$$

We get

$$
\begin{aligned}
\frac{d\mathbf{n}}{dt} &= -\alpha \oint d\theta \big[\hat{e}_x(\frac{\partial u_z}{\partial x}\cos^2\theta + \frac{\partial u_z}{\partial y}\cos\theta\sin\theta) + \hat{e}_y(\frac{\partial u_z}{\partial x}\sin\theta\cos\theta + \frac{\partial u_z}{\partial y}\sin^2\theta)\big] \\
&= -\alpha \oint d\theta \big[\hat{e}_x \cos^2\theta \frac{\partial u_z}{\partial x} + \hat{e}_y \sin^2\theta \frac{\partial u_z}{\partial y}\big] \\
&= -\alpha\pi \big[\hat{e}_x \frac{\partial u_z}{\partial x} + \hat{e}_y \frac{\partial u_z}{\partial y}\big] \\
&= -\alpha\pi(\nabla_\perp \mathbf{u}) \cdot \mathbf{n}
\end{aligned} \tag{13}
$$

It is reasonable to assume that $\alpha\pi = 1$ (sorry I don't know why) so

$$
\begin{aligned}
\frac{d\mathbf{n}}{dt} &= -\nabla_\perp \mathbf{u} \cdot \mathbf{n} \\
&= -[\nabla \mathbf{u} - \mathbf{n}(\mathbf{n} \cdot \nabla \mathbf{u})] \cdot \mathbf{n}
\end{aligned} \tag{14}
$$

3.3 Combine the two changes

Finally, let's combine Eq (6), (7) and (14) and get

$$\begin{aligned}\frac{d\mathbf{\Delta S}}{dt} &= \mathbf{n}\big[\Delta S(\nabla\cdot\mathbf{u}) - \mathbf{n}\cdot\nabla\mathbf{u}\cdot\mathbf{\Delta S}\big] - \Delta S[\nabla\mathbf{u} - \mathbf{n}(\mathbf{n}\cdot\nabla\mathbf{u})]\cdot\mathbf{n} \\ &= (\nabla\cdot\mathbf{u})\mathbf{\Delta S} - \nabla\mathbf{u}\cdot\mathbf{\Delta S}\end{aligned} \tag{15}$$

(the 2nd and the 4th terms counteract).

3.4 An indirect method

We can actually use the first two sections to indirectly derive $d\mathbf{\Delta S}/dt$. Write:

$$\Delta V = \mathbf{\Delta l}\cdot\mathbf{\Delta S} \tag{16}$$

Take time derivative and make use of Eq (2) and (4), we get:

$$(\nabla\cdot\mathbf{u})\mathbf{\Delta l}\cdot\mathbf{\Delta S} = (\mathbf{\Delta l}\cdot\nabla\mathbf{u})\cdot\mathbf{\Delta S} + \mathbf{\Delta l}\cdot\frac{d\mathbf{\Delta S}}{dt} \tag{17}$$

and we immediately get

$$\frac{d\mathbf{\Delta S}}{dt} = (\nabla\cdot\mathbf{u})\mathbf{\Delta S} - \nabla\mathbf{u}\cdot\mathbf{\Delta S} \tag{18}$$

4 Conclusion

In conclusion, the material derivatives of a line element $\mathbf{\Delta l}$, a surface element $\mathbf{\Delta S}$ and a volume element ΔV are:

$$\frac{d\mathbf{\Delta l}}{dt} = \mathbf{\Delta l}\cdot\nabla\mathbf{u} \tag{19a}$$

$$\frac{d\mathbf{\Delta S}}{dt} = (\nabla\cdot\mathbf{u})\mathbf{\Delta S} - \nabla\mathbf{u}\cdot\mathbf{\Delta S} \tag{19b}$$

$$\frac{d\Delta V}{dt} = (\nabla\cdot\mathbf{u})\Delta V \tag{19c}$$