Generalized Ohm's Law of a MHD plasma

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1 Derivation of the Generalized Ohm's Law

We start from the equations of motion for electron and ion (assumed to be proton):

$$\frac{\partial}{\partial t}(n_e m_e \mathbf{u_e}) + \nabla\cdot(n_e m_e \mathbf{u_e}\mathbf{u_e}) = -\nabla\cdot\mathbf{P_e} - e n_e(\mathbf{E} + \mathbf{u_e}\times\mathbf{B}) + \mathbf{R_e} \tag{1a}$$

$$\frac{\partial}{\partial t}(n_i m_i \mathbf{u_i}) + \nabla\cdot(n_i m_i \mathbf{u_i}\mathbf{u_i}) = -\nabla\cdot\mathbf{P_i} + e n_i(\mathbf{E} + \mathbf{u_i}\times\mathbf{B}) + \mathbf{R_i} \tag{1b}$$

Multiply the ion equation by m_e/m_i, then subtract the electron equation from the ion equation

$$m_e\frac{\partial}{\partial t}(n_i\mathbf{u_i} - n_e\mathbf{u_e}) + m_e\nabla\cdot(n_i\mathbf{u_i}\mathbf{u_i} - n_e\mathbf{u_e}\mathbf{u_e}) = \nabla\cdot(\mathbf{P_e} - \frac{m_e}{m_i}\mathbf{P_i}) + e\big[(n_e + \frac{m_e}{m_i}n_i)\mathbf{E} + (n_e\mathbf{u_e} + \frac{m_e}{m_i}n_i\mathbf{u_i})\times\mathbf{B}\big] - (\mathbf{R_e} - \frac{m_e}{m_i}\mathbf{R_i}) \tag{2}$$

where $\mathbf{R_i} = -\mathbf{R_e}$. Note that in MHD regime we have $n_e \approx n_i \approx n$ and we neglect all the $O(m_e/m_i)$ terms:

$$\frac{m_e}{e}\frac{\partial\mathbf{j}}{\partial t} + m_e\nabla\cdot[n(\mathbf{u_i}\mathbf{u_i} - \mathbf{u_e}\mathbf{u_e})] = \nabla\cdot\mathbf{P_e} + ne(\mathbf{E} + \mathbf{u_e}\times\mathbf{B}) - \mathbf{R_e} \tag{3}$$

We can then replace $\mathbf{u_e}$ by

$$\begin{aligned}\mathbf{u_e} &= \mathbf{u_i} - \frac{\mathbf{j}}{ne}\\ &\approx \mathbf{u} - \frac{\mathbf{j}}{ne}\end{aligned} \tag{4}$$

where $\mathbf{u}$ is the velocity of the plasma

$$\begin{aligned}\mathbf{u} &= \frac{m_i\mathbf{u_i} + m_e\mathbf{u_e}}{m_i + m_e}\\ &\approx \mathbf{u_i} + O(\frac{m_e}{m_i})\end{aligned} \tag{5}$$

and get

$$\mathbf{E} + \mathbf{u}\times\mathbf{B} = \frac{m_e}{ne^2}\frac{\partial\mathbf{j}}{\partial t} - \frac{1}{ne}\nabla\cdot\mathbf{P_e} + \frac{\mathbf{j}\times\mathbf{B}}{ne} + \frac{1}{ne}\mathbf{R_e} + \frac{m_e}{ne}\nabla\cdot[n(\mathbf{u_i}\mathbf{u_i} - \mathbf{u_e}\mathbf{u_e})] \tag{6}$$

The term $\mathbf{R_e}$ is the rate of change of the electron momentum due to the collision with ion, which can be written as

$$\begin{aligned}\mathbf{R_e} &= n m_e \nu_c(\mathbf{u_i} - \mathbf{u_e})\\ &= \frac{\nu_c m_e}{e}\mathbf{j}\end{aligned} \tag{7}$$

where ν_c is the collision frequency. By defining the resistivity

$$\eta = \frac{\nu_c m_e}{ne^2} \tag{8}$$

we can write the collision term as

$$\frac{\mathbf{R_e}}{ne} = \eta\mathbf{j} \tag{9}$$

Then, insert Eq (4) into the 2nd order term, we get

$$\frac{m_e}{ne}\nabla\cdot[n(\mathbf{u_i}\mathbf{u_i} - \mathbf{u_e}\mathbf{u_e})] = \frac{m_e}{ne^2}\nabla\cdot\big(\mathbf{u}\mathbf{j} + \mathbf{j}\mathbf{u} - \frac{\mathbf{j}\mathbf{j}}{ne}\big) \tag{10}$$

The final form of the **Generalized Ohm's Law** is:

$$\mathbf{E} + \mathbf{u}\times\mathbf{B} = \eta\mathbf{j} + \frac{1}{ne}\mathbf{j}\times\mathbf{B} - \frac{1}{ne}\nabla\cdot\mathbf{P_e} + \frac{m_e}{ne^2}\big[\frac{\partial\mathbf{j}}{\partial t} + \nabla\cdot\big(\mathbf{u}\mathbf{j} + \mathbf{j}\mathbf{u} - \frac{\mathbf{j}\mathbf{j}}{ne}\big)\big] \tag{11}$$

From Eq (8) it seems that the resistivity is proportional to $1/n$, however, note that the collision frequency ν_c is approximately proportional to n, η can be viewed as a constant.

2 Discussion on the R.H.S. terms

In practice, we can make some approximations to neglect some (or all) terms on the R.H.S. But first we must compare the magnitude of the different terms.

2.1 Hall term to resistive term

$$\frac{|\frac{1}{ne}\mathbf{j}\times\mathbf{B}|}{|\eta\mathbf{j}|} = \frac{\frac{JB}{ne}}{\frac{\nu_c m_e}{ne^2}J} = \frac{eB/m_e}{\nu_c} = \frac{\omega_{ge}}{\nu_c} \tag{12}$$

where ω_{ge} is the gyro-frequency of the electron.

2.2 Inertial term to resistive term

$$\frac{|\frac{m_e}{ne^2}\frac{\partial\mathbf{j}}{\partial t}|}{\eta\mathbf{j}} = \frac{\frac{m_e}{ne^2}\omega_J J}{\frac{\nu_c m_e}{ne^2}J} = \frac{\omega_J}{\nu_c} \tag{13}$$

where ω_J is the frequency of the variation of the current.

2.3 Inertial term to Hall term

It is then easy to get

$$\frac{|\frac{m_e}{ne^2}\frac{\partial\mathbf{j}}{\partial t}|}{|\frac{\mathbf{j}\times\mathbf{B}}{ne}|} = \frac{\omega_J}{\omega_{ge}} \tag{14}$$

2.4 Electron pressure term to resistive term

$$\frac{|\frac{1}{ne}\nabla\cdot\mathbf{P_e}|}{|\eta\mathbf{j}|} = \frac{\frac{nm_e v_{th,e}^2}{neL_p}}{\frac{\nu_c m_e}{ne^2}\frac{\Delta U}{ne}} = \frac{v_{th,e}}{\Delta U}\frac{v_{th,e}/L_p}{\nu_c} \tag{15}$$

where $v_{th,e}$ is the thermal velocity of the electron, L_p is the scale of variation of the electron pressure, ΔU is the difference between electron and ion velocities which causes the current.

2.5 Electron pressure term to Hall term

Similarly we can get

$$\frac{|\frac{1}{ne}\nabla\cdot\mathbf{P_e}|}{|\frac{\mathbf{j}\times\mathbf{B}}{ne}|} = \frac{v_{th,e}}{\Delta U}\frac{v_{th,e}/L_p}{\omega_{ge}} \tag{16}$$

2.6 Electron pressure term to inertial term

and

$$\frac{|\frac{1}{ne}\nabla\cdot\mathbf{P_e}|}{|\frac{m_e}{ne^2}\frac{\partial\mathbf{j}}{\partial t}|} = \frac{v_{th,e}}{\Delta U}\frac{v_{th,e}/L_p}{\omega_J} \tag{17}$$

2.7 Quadratic term

$$\frac{|\nabla\cdot\left(\mathbf{uj}+\mathbf{ju}-\frac{\mathbf{jj}}{ne}\right)|}{|\frac{\partial\mathbf{j}}{\partial t}|} = \frac{U/L_{qua}}{\omega_J} \tag{18}$$

where U is the characteristic speed of the plasma and L_{qua} is the length scale of the variation of the quadratic terms. Note that $\mathbf{jj}/ne$ is very small compared to $\mathbf{uj}$ and $\mathbf{ju}$ (on the order $\Delta U/U$).

2.8 Discussion

In general, whether we can neglect the resistive term, the Hall term or the inertial term depends on the relation between ν_c, ω_{ge} and ω_J. If the temperature of the electron is very small or L_p is very large, we can neglect the electron pressure term. Note that the quadratic terms may be of the same order of the inertial term.

Last, we need to point out that some times we can neglect all the R.H.S. terms if they are very small compared with the $\mathbf{u} \times \mathbf{B}$ term.