Frozen-in theorem

Chen Shi

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1 The frozen-in condition

The frozen-in condition of a MHD fluid is:

$$\mathbf{E} = -\mathbf{u} \times \mathbf{B} \tag{1}$$

2 Method 1: Derive directly

With the frozen-in condition (Eq (1)), it is easy to show that the magnetic flux through a surface moving with the fluid is time-invariant:

$$\begin{aligned}
\frac{d\Phi}{dt} &= \frac{d}{dt}\int \mathbf{B}\cdot\boldsymbol{\Delta}\mathbf{S} \\
&= \int \left\{\frac{d\mathbf{B}}{dt}\cdot\boldsymbol{\Delta}\mathbf{S} + \mathbf{B}\cdot\frac{d\boldsymbol{\Delta}\mathbf{S}}{dt}\right\} \\
&= \int \left\{(\frac{\partial\mathbf{B}}{\partial t} + \mathbf{u}\cdot\nabla\mathbf{B})\cdot\boldsymbol{\Delta}\mathbf{S} + \mathbf{B}\cdot[(\nabla\cdot\mathbf{u})\boldsymbol{\Delta}\mathbf{S} - \nabla\mathbf{u}\cdot\boldsymbol{\Delta}\mathbf{S}]\right\} \\
&= \int \boldsymbol{\Delta}\mathbf{S}\cdot\left\{\nabla\times(\mathbf{u}\times\mathbf{B}) + \mathbf{u}\cdot\nabla\mathbf{B} + (\nabla\cdot\mathbf{u})\mathbf{B} - \mathbf{B}\cdot\nabla\mathbf{u}\right\} \\
&= 0
\end{aligned} \tag{2}$$

where we have use the Faraday's law:

$$\frac{\partial\mathbf{B}}{\partial t} = -\nabla\times\mathbf{E} \tag{3}$$

and the material derivative of $\boldsymbol{\Delta}\mathbf{S}$

$$\frac{d\boldsymbol{\Delta}\mathbf{S}}{dt} = (\nabla\cdot\mathbf{u})\boldsymbol{\Delta}\mathbf{S} - \nabla\mathbf{u}\cdot\boldsymbol{\Delta}\mathbf{S} \tag{4}$$

3 Method 2: Derive using the magnetic potential

If we write the magnetic field in the vector potential $\mathbf{A}$ such that

$$\mathbf{B} = \nabla\times\mathbf{A} \tag{5}$$

then from the Faraday's law we get:

$$\nabla\times\frac{\partial\mathbf{A}}{\partial t} = \nabla\times(\mathbf{u}\times\mathbf{B}) \tag{6}$$

i.e.

$$\begin{aligned}
\frac{\partial\mathbf{A}}{\partial t} &= \mathbf{u}\times\nabla\times\mathbf{A} + \nabla\varphi \\
&= \nabla\mathbf{A}\cdot\mathbf{u} - \mathbf{u}\cdot\nabla\mathbf{A} + \nabla\varphi
\end{aligned} \tag{7}$$

We can then use the Stokes theorem to do the integral (note that $d\mathbf{\Delta l}/dt = \mathbf{\Delta l} \cdot \nabla \mathbf{u}$)

$$
\begin{aligned}
\frac{d}{dt}\int \mathbf{B} \cdot \mathbf{\Delta S} &= \frac{d}{dt}\oint \mathbf{\Delta l} \cdot \mathbf{A} \\
&= \oint \frac{d\mathbf{\Delta l}}{dt} \cdot \mathbf{A} + \mathbf{\Delta l} \cdot \frac{d\mathbf{A}}{dt} \\
&= \oint \mathbf{\Delta l} \cdot \left[\nabla \mathbf{u} \cdot \mathbf{A} + \frac{\partial \mathbf{A}}{\partial t} + \mathbf{u} \cdot \nabla \mathbf{A}\right] \\
&= \oint \mathbf{\Delta l} \cdot \left[\nabla \mathbf{u} \cdot \mathbf{A} + \nabla \mathbf{A} \cdot \mathbf{u} + \nabla \varphi\right] \\
&= \oint \mathbf{\Delta l} \cdot \nabla(\mathbf{u} \cdot \mathbf{A} + \varphi) \\
&= 0
\end{aligned}
\tag{8}
$$