Characteristic form of the linearized equation for adiabatic neutral gas

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1 Basic equations

The 1st order equations:

$$\frac{\partial \rho_1}{\partial t} + \mathbf{u_0} \cdot \nabla \rho_1 + \rho_0 \nabla \cdot \mathbf{u_1} + \mathbf{u_1} \cdot \nabla \rho_0 + \rho_1 \nabla \cdot \mathbf{u_0} = 0 \tag{1a}$$

$$\frac{\partial T_1}{\partial t} + \mathbf{u_0} \cdot \nabla T_1 + \mathbf{u_1} \cdot \nabla T_0 + (\gamma - 1)(\nabla \cdot \mathbf{u_0}) T_1 + (\gamma - 1)(\nabla \cdot \mathbf{u_1}) T_0 = 0 \tag{1b}$$

$$\frac{\partial \mathbf{u_1}}{\partial t} + \mathbf{u_0} \cdot \nabla \mathbf{u_1} + \mathbf{u_1} \cdot \nabla \mathbf{u_0} + \frac{1}{\rho_0} \nabla p_1 + \frac{\rho_1}{\rho_0} (\frac{\partial \mathbf{u_0}}{\partial t} + \mathbf{u_0} \cdot \nabla \mathbf{u_0}) = 0 \tag{1c}$$

As $p = \rho T$, we have

$$\nabla p_1 = \nabla(\rho_0 T_1 + \rho_1 T_0) = T_0 \nabla \rho_1 + \rho_1 \nabla T_0 + \rho_0 \nabla T_1 + T_1 \nabla \rho_0 \tag{2}$$

The variables are $\mathbf{U} = (\rho_1, T_1, u_{1x}, u_{1y}, u_{1z})$. We then write the equations as

$$\frac{\partial \rho_1}{\partial t} + \big[u_{0x} \frac{\partial \rho_1}{\partial x} + \rho_0 \frac{\partial u_{1x}}{\partial x}\big] + \big[u_{0y} \frac{\partial \rho_1}{\partial y} + \rho_0 \frac{\partial u_{1y}}{\partial y}\big] + \big[u_{0z} \frac{\partial \rho_1}{\partial z} + \rho_0 \frac{\partial u_{1z}}{\partial z}\big] + \big[(\nabla \cdot u_0)\rho_1 + \mathbf{u_1} \cdot \nabla \rho_0\big] = 0 \tag{3a}$$

$$\frac{\partial T_1}{\partial t} + \big[u_{0x} \frac{\partial T_1}{\partial x} + (\gamma-1) T_0 \frac{\partial u_{1x}}{\partial x}\big] + \big[u_{0y} \frac{\partial T_1}{\partial y} + (\gamma-1) T_0 \frac{\partial u_{1y}}{\partial y}\big] + \big[u_{0z} \frac{\partial T_1}{\partial z} + (\gamma-1) T_0 \frac{\partial u_{1z}}{\partial z}\big] + \big[\mathbf{u_1} \cdot \nabla T_0 + (\gamma-1)(\nabla \cdot \mathbf{u_0}) T_1\big] = 0 \tag{3b}$$

$$\begin{aligned} &\frac{\partial u_{1x}}{\partial t} + \big[u_{0x} \frac{\partial u_{1x}}{\partial x} + (\frac{T_0}{\rho_0} \frac{\partial \rho_1}{\partial x} + \frac{\partial T_1}{\partial x})\big] + \big[u_{0y} \frac{\partial u_{1x}}{\partial y}\big] + \big[u_{0z} \frac{\partial u_{1x}}{\partial z}\big] + \big[\frac{1}{\rho_0}(\frac{\partial T_0}{\partial x} \rho_1 + \frac{\partial \rho_0}{\partial x} T_1) \\ &+ \mathbf{u_1} \cdot \nabla u_{0x} + \frac{1}{\rho_0}(\frac{\partial u_{0x}}{\partial t} + \mathbf{u_0} \cdot \nabla u_{0x}) \rho_1\big] = 0 \end{aligned} \tag{3c}$$

$$\begin{aligned} &\frac{\partial u_{1y}}{\partial t} + \big[u_{0x} \frac{\partial u_{1y}}{\partial x}\big] + \big[u_{0y} \frac{\partial u_{1y}}{\partial y} + (\frac{T_0}{\rho_0} \frac{\partial \rho_1}{\partial y} + \frac{\partial T_1}{\partial y})\big] + \big[u_{0z} \frac{\partial u_{1y}}{\partial z}\big] + \big[\frac{1}{\rho_0}(\frac{\partial T_0}{\partial y} \rho_1 + \frac{\partial \rho_0}{\partial y} T_1) \\ &+ \mathbf{u_1} \cdot \nabla u_{0y} + \frac{1}{\rho_0}(\frac{\partial u_{0y}}{\partial t} + \mathbf{u_0} \cdot \nabla u_{0y}) \rho_1\big] = 0 \end{aligned} \tag{3d}$$

$$\begin{aligned} &\frac{\partial u_{1z}}{\partial t} + \big[u_{0x} \frac{\partial u_{1z}}{\partial x}\big] + \big[u_{0y} \frac{\partial u_{1z}}{\partial y}\big] + \big[u_{0z} \frac{\partial u_{1z}}{\partial z} + (\frac{T_0}{\rho_0} \frac{\partial \rho_1}{\partial z} + \frac{\partial T_1}{\partial z})\big] + \big[\frac{R}{\rho_0}(\frac{\partial T_0}{\partial z} \rho_1 + \frac{\partial \rho_0}{\partial z} T_1) \\ &+ \mathbf{u_1} \cdot \nabla u_{0z} + \frac{1}{\rho_0}(\frac{\partial u_{0z}}{\partial t} + \cdot \mathbf{u_0} \cdot \nabla u_{0z}) \rho_1\big] \end{aligned} \tag{3e}$$

In a more compact form:

$$\frac{\partial \mathbf{U}}{\partial t} + \mathbf{A} \frac{\partial \mathbf{U}}{\partial x} + \mathbf{C} \frac{\partial \mathbf{U}}{\partial y} + \mathbf{D} \frac{\partial \mathbf{U}}{\partial z} + \mathbf{F} = 0 \tag{4}$$

and (take A as an example)

$$\mathbf{A} = \begin{pmatrix} u_{0x} & 0 & \rho_0 & 0 & 0 \\ 0 & u_{0x} & (\gamma-1)T_0 & 0 & 0 \\ \frac{T_0}{\rho_0} & 1 & u_{0x} & 0 & 0 \\ 0 & 0 & 0 & u_{0x} & 0 \\ 0 & 0 & 0 & 0 & u_{0x} \end{pmatrix} \tag{5}$$

$\mathbf{F}$ includes the other terms without any derivatives (of 1st order quantities).

2 Decomposition of the coefficient matrices

For $\mathbf{A}$ (waves propagating along x), we can calculate the eigenvalues of it:

$$u_{0x}, u_{0x}, u_{0x}, u_{0x} - c_s, u_{0x} + c_s$$

where $c_s = \sqrt{\gamma T_0}$. It is obvious that u_{1y} and u_{1z} are only convected by the mean flow u_{0x}, i.e., there are no transverse waves in the gas. Thus we can take just the top left 3×3 block of $\mathbf{A}$ to do the following calculation. We write:

$$\mathbf{A} = \mathbf{S \Lambda S^{-1}}$$

where

$$\mathbf{\Lambda} = \begin{pmatrix} u_{0x} & 0 & 0 \\ 0 & u_{0x} - c_s & 0 \\ 0 & 0 & u_{0x} + c_s \end{pmatrix} \tag{6}$$

and $\mathbf{S}$ is the matrix whose columns are the eigenvectors of $\mathbf{A}$:

$$\mathbf{S} = \begin{pmatrix} -\frac{\rho_0}{T_0} & -\frac{\rho_0}{c_s} & \frac{\rho_0}{c_s} \\ 1 & (\frac{1}{\gamma} - 1)c_s & (1 - \frac{1}{\gamma})c_s \\ 0 & 1 & 1 \end{pmatrix} \tag{7}$$

and $\mathbf{S^{-1}}$ is the inverse of $\mathbf{S}$ (whose rows are the left eigenvectors of $\mathbf{A}$):

$$\mathbf{S^{-1}} = \begin{pmatrix} \frac{T_0}{\rho_0}(\frac{1}{\gamma} - 1) & \frac{1}{\gamma} & 0 \\ -\frac{c_s}{2\gamma\rho_0} & -\frac{1}{2c_s} & \frac{1}{2} \\ \frac{c_s}{2\gamma\rho_0} & \frac{1}{2c_s} & \frac{1}{2} \end{pmatrix} \tag{8}$$

If we write $\mathbf{S} = (\mathbf{r_1}, \mathbf{r_2}, \mathbf{r_3})$ and $\mathbf{S^{-1}} = (\mathbf{l_1}, \mathbf{l_2}, \mathbf{l_3})^T$, we can then decompose the term $\mathbf{A}\frac{\partial \mathbf{U}}{\partial x}$ as:

$$\begin{aligned} (\mathbf{A}\frac{\partial \mathbf{U}}{\partial x})_i &= \sum_{j=1}^{3} \big[(\mathbf{S \Lambda S^{-1}})_{ij} \frac{\partial U_j}{\partial x}\big] \\ &= \sum_{j=1}^{3} \frac{\partial U_j}{\partial x} \sum_{k=1}^{3} \lambda_k S_{ik} S^{-1}_{kj} \\ &= \sum_{k=1}^{3} \lambda_k r_{ki} \sum_{j=1}^{3} l_{kj} \frac{\partial U_j}{\partial x} \end{aligned} \tag{9}$$

From Eq (9), we immediately see that we can define 3 characteristics:

$$\begin{aligned} \mathcal{L}_0 &= u_{0x} \sum_{j=1}^{3} l_{1j} \frac{\partial U_j}{\partial x} \\ &= u_{0x} \big[\frac{T_0}{\rho_0}(\frac{1}{\gamma} - 1)\frac{\partial \rho_1}{\partial x} + \frac{1}{\gamma}\frac{\partial T_1}{\partial x}\big] \end{aligned} \tag{10a}$$

$$\begin{aligned} \mathcal{L}_- &= (u_{0x} - c_s) \sum_{j=1}^{3} l_{2j} \frac{\partial U_j}{\partial x} \\ &= (u_{0x} - c_s)\big[\frac{1}{2}\frac{\partial u_{1x}}{\partial x} - \frac{1}{2c_s}(\frac{T_0}{\rho_0}\frac{\partial \rho_1}{\partial x} + \frac{\partial T_1}{\partial x})\big] \end{aligned} \tag{10b}$$

$$\mathcal{L}_+ = (u_{0x} + c_s)\sum_{j=1}^{3} l_{3j}\frac{\partial U_j}{\partial x}$$
$$= (u_{0x} + c_s)\big[\frac{1}{2}\frac{\partial u_{1x}}{\partial x} + \frac{1}{2c_s}(\frac{T_0}{\rho_0}\frac{\partial \rho_1}{\partial x} + \frac{\partial T_1}{\partial x})\big] \quad (10c)$$

and write the x-derivatives of ρ_1, T_1, u_{1x} as:

$$\frac{\partial \rho_1}{\partial t} = \frac{\rho_0}{T_0}\mathcal{L}_0 - \frac{\rho_0}{c_s}(\mathcal{L}_+ - \mathcal{L}_-)$$
$$\frac{\partial T_1}{\partial t} = -\mathcal{L}_0 - (1 - \frac{1}{\gamma})c_s(\mathcal{L}_+ - \mathcal{L}_-)$$
$$\frac{\partial u_{1x}}{\partial t} = -(\mathcal{L}_+ + \mathcal{L}_-)$$

where $\mathcal{L}_0$, $\mathcal{L}_-$, $\mathcal{L}_+$ correspond to entropy mode and the two sound waves (in opposite directions). We can then easily get the results in y and z through the coordinates transformation: $(x, y, z) \rightarrow (y, z, x)$.