Basics on turbulence study, the Kolmogorov's theory and its applications

Chen Shi

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1 Basic concepts

In studying turbulence, we typically assume an infinite space (or a periodic box) such that the quantities can be Fourier transformed (assume 1D and neglect the possible coefficient $1/2\pi$ for simplicity):

$$\hat{v}(k) = \int_{-\infty}^{\infty} v(x) e^{-ikx} dx, \quad v(x) = \int_{-\infty}^{\infty} \hat{v}(k) e^{ikx} dk \tag{1}$$

One important theorem is the ***Parseval's theorem***:

$$\int f g^* dx = \int \hat{f} \hat{g}^* dk \tag{2}$$

Here f and g are two square-integrable functions and the superscript $*$ represents conjugation. One immediate relation is acquired by set $g = f$:

$$\int |f|^2 dx = \int \left|\hat{f}\right|^2 dk \tag{3}$$

i.e. the total energy in the original function f equals the total energy in its Fourier amplitude $\hat{f}$.

Then we introduce the space-averaging operator $\langle\rangle$

$$\langle v(x) \rangle = \lim_{L\to\infty} \frac{1}{L} \int_{-L/2}^{L/2} v(x) dx \tag{4}$$

And the ***low-pass filter***:

$$v_k^<(x) = \int_{-k}^{k} \hat{v}(k') e^{ik'x} dk' \tag{5}$$

which means keeping only the modes with wave vector $|k'| < k$. Then we are able to to define the ***cumulative energy spectrum*** $\mathcal{E}(k)$:

$$\mathcal{E}(k) = \left\langle \left| v_k^<(x) \right|^2 \right\rangle \tag{6}$$

Intuition tells us that $\mathcal{E}(k)$ represents the energy in all the modes with scales larger than $1/k$ but we need more rigorous proof. Bcause $v_\infty^<(x) = v(x)$, we get

$$\mathcal{E}(k \to \infty) = \left\langle |v(x)|^2 \right\rangle \tag{7}$$

We now prove that that the above relation is actually the Parseval's theorem Eq (3):

$$\begin{aligned} \mathcal{E}(k) &= \left\langle \left| v_k^<(x) \right|^2 \right\rangle \\ &= \left\langle \int_{-k}^{k} \hat{v}(k_1) e^{ik_1 x} dk_1 \times \int_{-k}^{k} \hat{v}^*(k_2) e^{-ik_2 x} dk_2 \right\rangle \end{aligned}$$

$$
\begin{aligned}
&= \left\langle \int_{-k}^{k} \int_{-k}^{k} \hat{v}(k_1)\hat{v}^*(k_2) e^{i(k_1-k_2)x} dk_1 dk_2 \right\rangle \\
&= \lim_{L\to\infty} \frac{1}{L} \int_{-L/2}^{L/2} \int_{-k}^{k} \int_{-k}^{k} \hat{v}(k_1)\hat{v}^*(k_2) e^{i(k_1-k_2)x} dk_1 dk_2 dx \\
&= \int_{-k}^{k} \int_{-k}^{k} \hat{v}(k_1)\hat{v}^*(k_2) \left[\lim_{L\to\infty} \frac{1}{L} \int_{-L/2}^{L/2} e^{i(k_1-k_2)x} dx \right] dk_1 dk_2 \\
&= \int_{-k}^{k} \int_{-k}^{k} \hat{v}(k_1)\hat{v}^*(k_2) \left[\lim_{L\to\infty} \frac{1}{L} \delta(k_1-k_2) \right] dk_1 dk_2 \\
&= \lim_{L\to\infty} \frac{1}{L} \int_{-k}^{k} \hat{v}(k_1) \left[\int_{-k}^{k} \hat{v}^*(k_2)\delta(k_1-k_2) dk_2 \right] dk_1
\end{aligned}
$$

Note that k_1 and k_2 are both in the range $[-k, k]$ thus the integral

$$
\int_{-k}^{k} \hat{v}^*(k_2)\delta(k_1-k_2) dk_2 = \hat{v}^*(k_1)
$$

Then we get

$$
\mathcal{E}(k) = \lim_{L\to\infty} \frac{1}{L} \int_{-k}^{k} \left|\hat{v}(k_1)\right|^2 dk_1 \tag{8}
$$

From Eq (8), as we have stated before, the physical meaning of $\mathcal{E}(k)$ is *the energy in all the modes with scales larger than* $1/k$. If we take the limit $k \to \infty$ and make use of Eq (7), we get

$$
\mathcal{E}(k\to\infty) = \lim_{L\to\infty} \frac{1}{L} \int_{-\infty}^{\infty} \left|\hat{v}(k_1)\right|^2 dk_1 = \left\langle |v(x)|^2 \right\rangle = \lim_{L\to\infty} \frac{1}{L} \int_{-\infty}^{\infty} |v(x)|^2 dx \tag{9}
$$

which gives the *Parseval's theorem* Eq (3).

Then we introduce the very import quantity, ***energy spectrum*** $E(k)$

$$
E(k) = \frac{d\mathcal{E}(k)}{dk} \tag{10}
$$

which gives (notice that $\mathcal{E}(0) = 0$)

$$
\int_0^k E(k')dk' = \mathcal{E}(k) \tag{11}
$$

From Eq (8), we get

$$
E(k) = \lim_{L\to\infty} \frac{1}{L} \left[|\hat{v}(k)|^2 + |\hat{v}(-k)|^2 \right] \tag{12}
$$

i.e. $E(k)$ is *the energy in the Fourier modes* k *and* $-k$. Last, a relation which will be frequently used in the semi-quantitative analysis of turbulence, e.g. Kolmogorov's theory, is the estimate of the perturbation amplitude at scale $1/k$:

$$
v(k) \sim \sqrt{kE(k)} \tag{13}
$$

Eq (13) is merely a dimensional analysis: notice that $\hat{v}(k)$ is of the dimension v/k so the dimension of $E(k) \sim k\hat{v}^2 \sim v^2/k$, which gives the estimate Eq (13).

2 Kolmogorov's theory

In this note, we are not going to present the complete and rigorous theory of Kolmogorov but only the most basic ideas. We first write down the Navier-Stokes equation

$$
\frac{\partial \mathbf{v}}{\partial t} + \mathbf{v}\cdot\nabla\mathbf{v} = -\frac{1}{\rho}\nabla p + \frac{1}{R}\nabla^2\mathbf{v} \tag{14}
$$

The only term that controls the turbulence is the nonlinear term $\mathbf{v} \cdot \nabla \mathbf{v}$. When the Reynolds number R is very large, the energy is only able to dissipate at very small scale λ_d, which is called the *dissipation scale*, at which the Reynolds number becomes as small as $O(1)$. On the other hand, energy is only injected at very large scales λ_e, which is called the *energy-containing scale*, e.g. the scale of the bulk flow. What we care most is the scale far from the dissipation and energy-containing scales

$$\lambda_e \gg \lambda \gg \lambda_d \tag{15}$$

We call this range of scales the *inertial range*. At these scales, energy is either dissipated nor injected locally. As a result, energy is only allowed to flow through them. That is to say, at all these scales, the energy dissipation rate is the same and equal to that at λ_d. We write this energy dissipation rate as ε. Physically, ε is the rate of dissipation of the energy per unit mass so it has the dimension $\varepsilon \sim v^2/\tau_{NL}$. The key step in the analysis of any turbulence is to determine τ_{NL}, i.e. the ***nonlinear interaction time***, sometimes called the eddy turnover time.

In the incompressible flow, τ_{NL} can be easily estimated by comparing the two terms on the L.H.S. of Eq (14):

$$\frac{v}{\tau_{NL}} \sim \frac{v^2}{\lambda} \tag{16}$$

which gives

$$\tau_{NL} \sim \frac{\lambda}{v} \tag{17}$$

and thus

$$\varepsilon \sim v^3/\lambda \tag{18}$$

We then make use Eq (13) and $\lambda \sim 1/k$ to get

$$\epsilon \sim [kE(k)]^{\frac{3}{2}}\, k \tag{19}$$

which then gives the famous Kolmogorov's $-5/3$ law

$$E(k) \sim \varepsilon^{\frac{2}{3}} k^{-\frac{5}{3}} \tag{20}$$

3 Applications

3.1 Kraichnan's theory of Alfvénic turbulence

We then apply the same analysis to the Alfvénic turbulence, as first described by (Kraichnan, 1965). For Alfvénic turbulence, incompressibility is assumed and we write the magnetic field in Alfvén speed unit $b = B/\sqrt{\mu_0 \rho}$. Equipartition of the magnetic and kinetic energies is also assumed (for pure Alfvénic waves this is true but in the solar wind this is not true!):

$$v(k) \sim b(k) \sim \sqrt{kE(k)} \tag{21}$$

In the pure hydrodynamic case, Eq (17) tells us that at scale λ, the nonlinear time is just the time for the velocity perturbation at this scale going through the length λ. In the Alfvénic turbulence, there exists a strong enough background magnetic field b_0 and the nonlinear interaction happens between the Alfvén waves propagating in opposite directions. Thus, for waves with wavelength λ, the nonlinear interaction time is the time for the waves to move one wavelength λ:

$$\tau_{NL} \sim \frac{\lambda}{b_0} \tag{22}$$

Note that on the denominator is the background magnetic field rather than the magnetic field on a specific scale so that this nonlinear time is much smaller than the time $\lambda/b(k)$. The background field controls the energy dissipation rate: the stronger b_0, the faster the dissipation, i.e. $\varepsilon \propto \tau_{NL}$. By dimensional analysis:

$$\frac{\lambda}{b_0} E^\alpha k^\beta \sim \varepsilon \sim \frac{b^3}{\lambda} \tag{23}$$

we get $\alpha = 2, \beta = 4$ and then the Kraichnan's $-3/2$ law:

$$E(k) \sim (b_0 \varepsilon)^{1/2} k^{-\frac{3}{2}} \tag{24}$$

3.2 Hall-MHD turbulence

Please refer to (Galtier and Buchlin, 2018) for more details. In Hall-MHD, the normalized induction equation is

$$\frac{\partial \mathbf{b}}{\partial t} = \nabla \times (\mathbf{u} \times \mathbf{b}) + \frac{1}{S} \nabla^2 \mathbf{b} - d_i \nabla \times \left(\frac{(\nabla \times \mathbf{b}) \times \mathbf{b}}{\rho} \right) \tag{25}$$

When the ion inertial length is large (i.e. the scales are are considering is comparable or smaller than d_i), the nonlinear interaction comes from the Hall term. We can estimate the nonlinear time (still write the magnetic field in speed unit)

$$\frac{b}{\tau_{NL}} \sim \frac{d_i b^2}{\lambda^2} \tag{26}$$

i.e.

$$\tau_{NL} = \frac{\lambda^2}{d_i b} \tag{27}$$

Then we have two choices: (1) the energy is magnetic-dominated and (2) the energy is kinetic-dominated. For case (1), the magnetic energy b^2 is dissipated, i.e.

$$\varepsilon \sim \frac{b^2}{\tau_{NL}} \sim d_i \frac{b^3}{\lambda^2} \tag{28}$$

Plug in $b \sim \sqrt{E_b(k) k}$, we get

$$E_b(k) \sim \left(\frac{\varepsilon}{d_i} \right)^{\frac{2}{3}} k^{-7/3} \tag{29}$$

From the momentum equation

$$\frac{\partial \mathbf{v}}{\partial t} + \mathbf{v} \cdot \nabla \mathbf{v} = -\frac{1}{\rho} \nabla p + \frac{1}{R} \nabla^2 \mathbf{v} + \frac{1}{\rho} (\nabla \times \mathbf{b}) \times \mathbf{b} \tag{30}$$

by equating the time-derivative term and the magnetic term, we get

$$\frac{u}{\tau_{NL}} \sim \frac{b^2}{\lambda} \tag{31}$$

i.e.

$$\tau_{NL} = \frac{\lambda u}{b^2} \tag{32}$$

Equating the above equation with Eq (27) gives

$$u \sim \frac{\lambda b}{d_i} \tag{33}$$

i.e.

$$E_u(k) \sim d_i^{-2} k^{-2} E_b \sim \varepsilon^{2/3} d_i^{-8/3} k^{-13/3} \tag{34}$$

For case (2), the kinetic energy is dissipated, i.e.

$$\varepsilon \sim \frac{u^2}{\tau_{NL}} \tag{35}$$

As the momentum equation is dominated by kinetic energy, so the nonlinear time is essentially the same as the Kolmogorov's one:

$$\tau_{NL} \sim \frac{\lambda}{u} \tag{36}$$

and the spectrum of velocity is also Kolmogorov:

$$E_u(k) \sim \varepsilon^{\frac{2}{3}} k^{-\frac{5}{3}} \tag{37}$$

By equating the nonlinear time with Eq (27), we get

$$u \sim d_i b/\lambda \tag{38}$$

i.e.

$$E_b(k) \sim \varepsilon^{2/3} d_i^{-2} k^{-11/3} \tag{39}$$