Tearing Mode Instability in Magnetohydrodynamic Current Sheets

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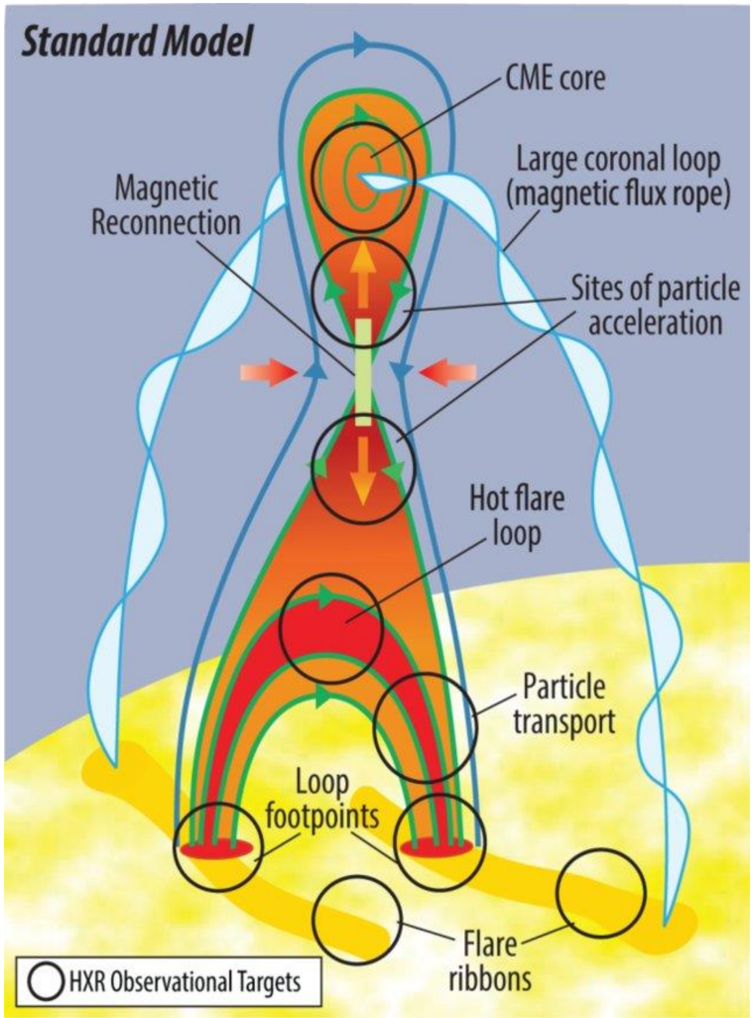
Outline

- Introduction—onset problem of reconnection
- Basic equations
- Boundary-layer problem
- Key results for the simplest 1D case
- 2D current sheet: aspect ratio matters
- Other effects: viscosity, Hall effect, guide field, etc.
- Conclusion

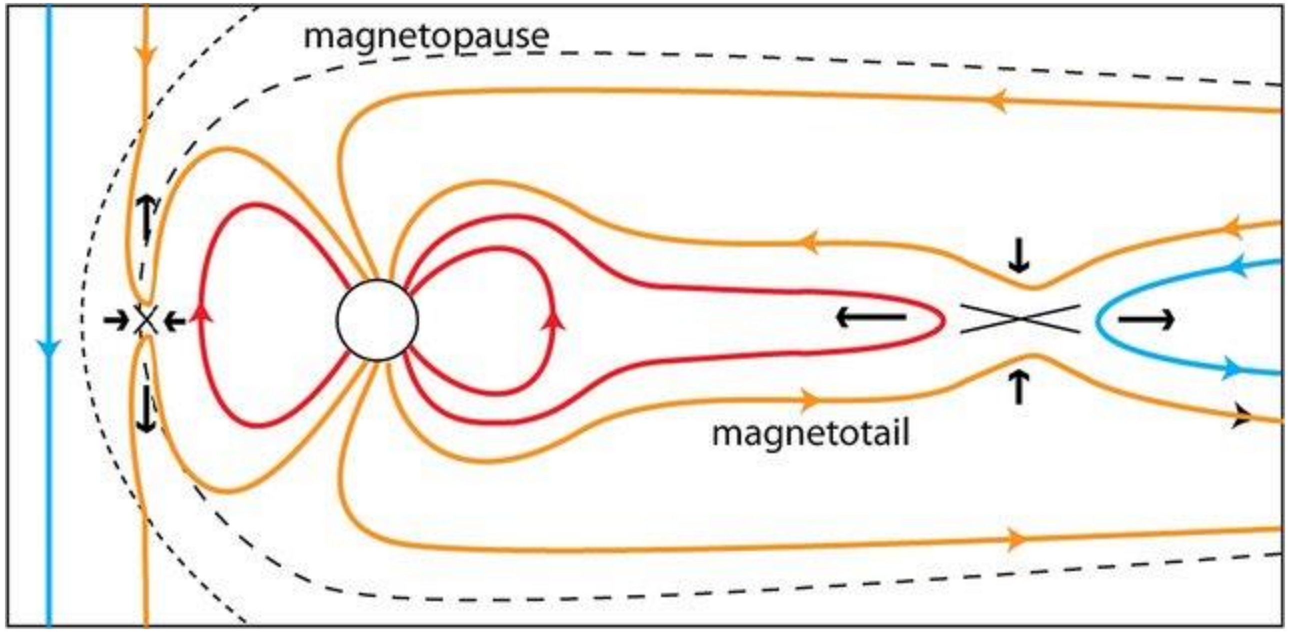
Magnetic reconnection—*explosive energy release*

CME and solar flare



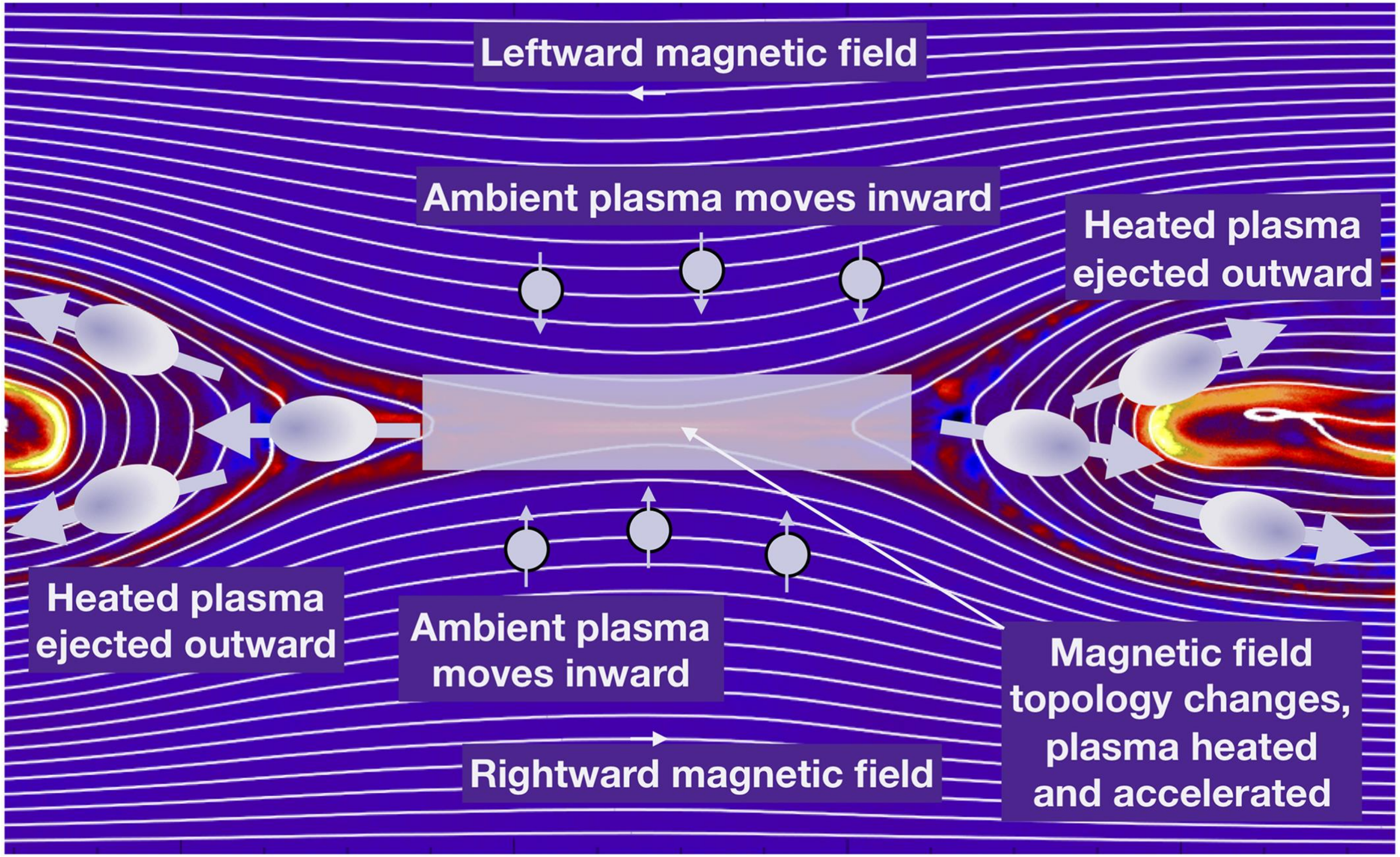
Christe et al., 2017

substorm



Eastwood et al., 2017

Steadily-reconnecting current sheet



Hesse and Cassak, 2019

What triggers the fast magnetic reconnection? → at MHD scales…

Onset of reconnection

Ideal MHD does not allow "reconnection" of field lines

- Need to break the "frozen-in" condition
- Generalized Ohm's law:

$$\boxed{\boldsymbol{E} + \boldsymbol{u} \times \boldsymbol{B} = \eta \boldsymbol{J}} + \frac{\boldsymbol{J} \times \boldsymbol{B}}{ne} - \frac{\boldsymbol{\nabla} \cdot \boldsymbol{P_e}}{ne} + \frac{m_e}{ne^2} \frac{\partial \boldsymbol{J}}{\partial t}$$

resistivity   Hall   Electron pressure   Electron inertia

Resistivity can be induced by collisions or wave-particle interactions

Normalization of (viscous-resistive)-MHD equations

$$\frac{\partial \rho}{\partial t} + \nabla \cdot (\rho \boldsymbol{V}) = 0$$

$$\rho\left(\frac{\partial \boldsymbol{V}}{\partial t} + \boldsymbol{V} \cdot \nabla \boldsymbol{V}\right) = -\nabla P + \boldsymbol{J} \times \boldsymbol{B} + \nu\rho\nabla^2 \boldsymbol{V}$$

or $\nabla \cdot (\nu\rho\nabla\boldsymbol{V})$

$$\frac{\partial \boldsymbol{B}}{\partial t} = \nabla \times (\boldsymbol{V} \times \boldsymbol{B}) + \eta\nabla^2 \boldsymbol{B}$$

$$\frac{d(P\rho^{-\kappa})}{dt} = 0$$

Here we have neglected μ_0

MHD equation, like the Navier-Stokes equation, can be **non-dimensionalized**.

Choose a set of reference values: $(\bar{\rho}, \bar{B}, L)$ ← They are arbitrary

From the three values, we can further define reference speed and time:

$$\bar{V} = \frac{\bar{B}}{\sqrt{\mu_0 \bar{\rho}}}$$

$$\tau = L/\bar{V}$$

and pressure:

$$\bar{P} = \bar{\rho}\bar{V}^2 = \bar{B}^2/\mu_0$$

We can then normalize all the fields to the above values, e.g.,

$$\rho = \bar{\rho}\tilde{\rho}, \boldsymbol{V} = \bar{V}\widetilde{\boldsymbol{V}}$$

Here the "$\sim$" values are normalized fields.

Note that the derivatives also need to be normalized:

$$\frac{\partial}{\partial t} = \frac{1}{\tau}\frac{\partial}{\partial \tilde{t}}, \qquad \nabla = \frac{1}{L}\tilde{\nabla}$$

The MHD equation set becomes:

$$\frac{\partial \tilde{\rho}}{\partial \tilde{t}} + \tilde{\nabla}\cdot\left(\tilde{\rho}\tilde{\boldsymbol{V}}\right) = 0$$

$$\tilde{\rho}\left(\frac{\partial \tilde{\boldsymbol{V}}}{\partial \tilde{t}} + \tilde{\boldsymbol{V}}\cdot\tilde{\nabla}\tilde{\boldsymbol{V}}\right) = -\tilde{\nabla}\tilde{P} + \tilde{\boldsymbol{J}}\times\tilde{\boldsymbol{B}} + \boxed{\frac{1}{R}}\tilde{\rho}\tilde{\nabla}^2\tilde{\boldsymbol{V}}$$

$$\frac{\partial \tilde{\boldsymbol{B}}}{\partial \tilde{t}} = \tilde{\nabla}\times\left(\tilde{\boldsymbol{V}}\times\tilde{\boldsymbol{B}}\right) + \boxed{\frac{1}{S}}\tilde{\nabla}^2\tilde{\boldsymbol{B}}$$

$$\frac{d\left(\tilde{P}\tilde{\rho}^{-\kappa}\right)}{d\tilde{t}} = 0$$

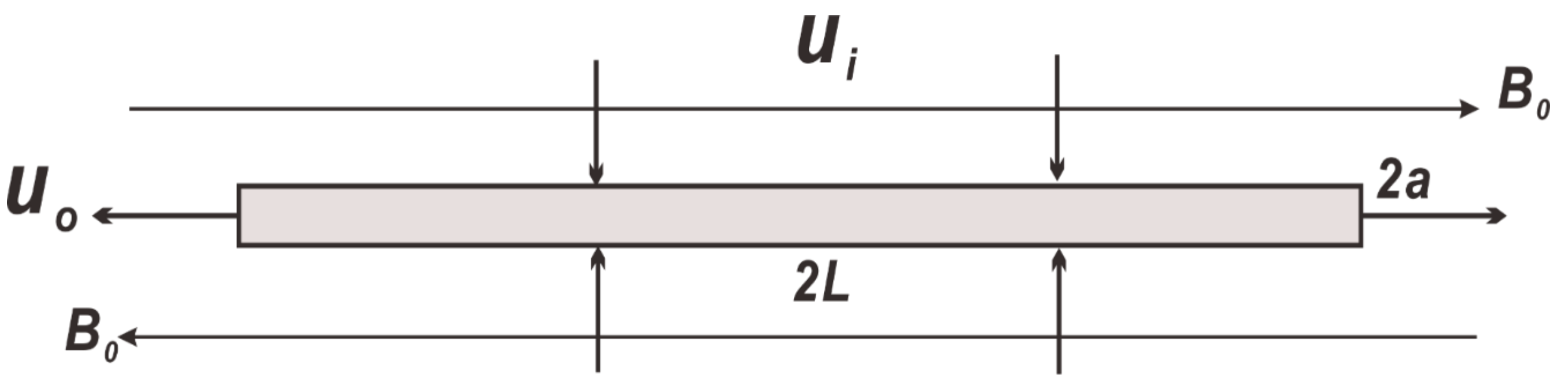
Reynolds number $R = \frac{L\bar{V}}{\nu}$

Lundquist number $S = \frac{L\bar{V}}{\eta}$

a.k.a. magnetic Reynolds number

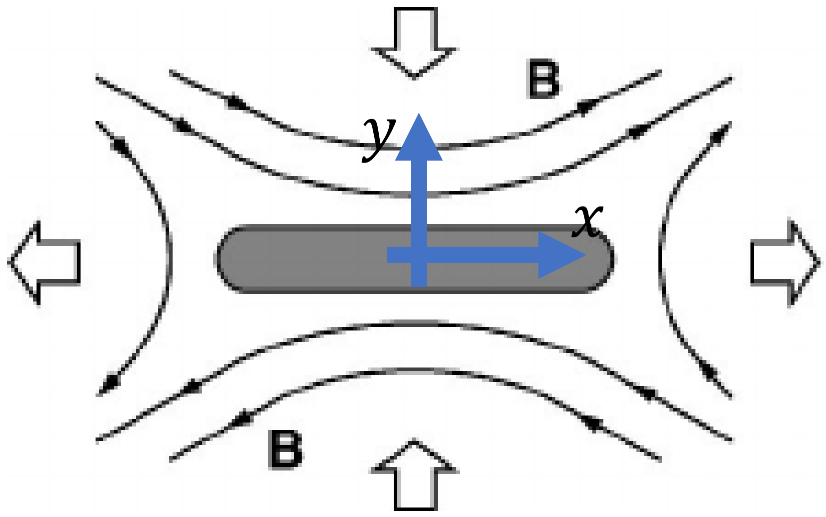
Exercise: show that the (magnetic) Reynolds number is actually τ_d/τ_c where τ_d is the diffusion time scale, τ_c is the convection time scale

Sweet-Parker model



Sweet-Parker current sheet

Ji et al., 1999



$$P^T = Constant$$

$$\boldsymbol{V} \cdot \nabla \boldsymbol{V} = \frac{1}{\mu_0 \rho} \boldsymbol{B} \cdot \nabla \boldsymbol{B}$$

Along the central vertical line ($x = 0$):
$V_x = B_y \equiv 0, V_y \neq 0, B_x \neq 0$:

$$V_y \frac{\partial \boldsymbol{V}}{\partial y} = \frac{1}{\mu_0 \rho} B_x \frac{\partial \boldsymbol{B}}{\partial x}$$

x direction is exactly 0 $\left(\frac{\partial B_x}{\partial x} \equiv 0\right)$

y direction: $V_y \frac{\partial V_y}{\partial y} = \frac{1}{\mu_0 \rho} B_x \frac{\partial B_y}{\partial x}$

$$\boxed{\frac{V_y^2}{a} = \frac{B_x B_y}{\mu_0 \rho L}}$$

These quantities are averaged values

$$\boxed{\frac{V_x^2}{L} = \frac{B_x B_y}{\mu_0 \rho a}}$$

Along the central horizontal line ($y = 0$):
$V_y = B_x \equiv 0, V_x \neq 0, B_y \neq 0$:

$$V_x \frac{\partial \boldsymbol{V}}{\partial x} = \frac{1}{\mu_0 \rho} B_y \frac{\partial \boldsymbol{B}}{\partial y}$$

y direction is exactly 0 $\left(\frac{\partial B_y}{\partial x} \equiv 0\right)$

x direction: $V_x \frac{\partial V_x}{\partial x} = \frac{1}{\mu_0 \rho} B_y \frac{\partial B_x}{\partial y}$

Incompressibility:

$$\nabla \cdot \boldsymbol{V} = 0$$

$$\frac{\partial V_x}{\partial x} + \frac{\partial V_y}{\partial y} = 0$$

$$\boxed{\frac{V_x}{L} \sim \frac{V_y}{a}}$$

$$\boxed{\frac{V_x^2}{L} = \frac{B_x B_y}{\mu_0 \rho a}}$$

Similarly:

$$\nabla \cdot \boldsymbol{B} = 0$$

$$\boxed{\frac{B_x}{L} \sim \frac{B_y}{a}}$$

$$\boxed{V_x^2 \sim \frac{B_x^2}{\mu_0 \rho}}$$

$$\boxed{\frac{V_y^2}{a} = \frac{B_x B_y}{\mu_0 \rho L}}$$

$$\boxed{V_y^2 \sim \frac{B_y^2}{\mu_0 \rho}}$$

$$\nabla \times (\boldsymbol{V} \times \boldsymbol{B}) = \boldsymbol{B} \cdot \nabla \boldsymbol{V} - \boldsymbol{V} \cdot \nabla \boldsymbol{B}$$

$$\boldsymbol{B} \cdot \nabla \boldsymbol{V} - \boldsymbol{V} \cdot \nabla \boldsymbol{B} + \eta \nabla^2 \boldsymbol{B} = 0$$

$$\nabla^2 = \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} \approx \frac{\partial^2}{\partial y^2} \approx \frac{1}{a^2}$$

Along the central vertical line ($x = 0$) apply a similar analysis, we get:

$$V_y \sim \frac{\eta}{a}$$

Convection balances diffusion of $\boldsymbol{B}$

Thus, we also have:

$$V_x \sim \frac{\eta L}{a^2} \sim V_A$$

Remind: $S = LV_A/\eta$

V_A: Upstream Alfvén speed

$$\boxed{\frac{a}{L} \sim S^{-1/2}}$$

Reconnection rate:

$$R = V_y B_x = V_x B_x \frac{a}{L} \propto S^{-1/2}$$

SLOW!

Are resistive current sheets stable?

- A current sheet has free energy…
- resistivity can help release the energy
- Sweet-Parker model is a ***stable*** way to convert magnetic energy to kinetic energy through diffusion (resistivity)
- However, ***instability*** also exists in a resistive current sheet— ***tearing mode instability*** which can release energy faster!

(Furth et al., 1963, PhFl)

Linearize the MHD equation set

Background fields:

$$\rho = Const$$
$$P^T = Const$$
$$\boldsymbol{V} \equiv 0$$
$$\boldsymbol{B} = B_x(y)\,\hat{e}_x + B_z(y)\hat{e}_z$$

1D configuration: infinite along x direction, the fields are functions of y only.

$\boldsymbol{B} \cdot \nabla \boldsymbol{B} \equiv 0 \longrightarrow$ Momentum equation is satisfied

Faraday equation is not satisfied: $\boldsymbol{E} = -\boldsymbol{V} \times \boldsymbol{B} + \eta \boldsymbol{J} = \eta \boldsymbol{J}$, thus we have:

$$\frac{\partial \boldsymbol{B}}{\partial t} = \eta \nabla^2 \boldsymbol{B}$$

There is diffusion of the background magnetic field. But we only discuss the very small resistivity case, so the diffusion is slow compared with the tearing instability

First-order fields: $\rho_1 \equiv 0$ (assume incompressibility), $p, \boldsymbol{u}, \boldsymbol{b}$ and we write any perturbation in the form

$$\boxed{f(x, y, z, t) = f(y)\exp(ikx + \gamma t)}$$

This is equivalent to apply a Fourier transform in x and a Fourier (Laplace) transform in t to the linearized equation set

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Momentum equation:

$$\rho\frac{\partial \boldsymbol{V}}{\partial t} + \rho\boldsymbol{V}\cdot\nabla\boldsymbol{V} = -\nabla\left(P + \frac{B^2}{2\mu_0}\right) + \frac{1}{\mu_0}\boldsymbol{B}\cdot\nabla\boldsymbol{B}$$

$$\rho\gamma\boldsymbol{u} \qquad 0 \qquad p \qquad \frac{\boldsymbol{B}\cdot\boldsymbol{b}}{\mu_0} \qquad \frac{\boldsymbol{B}\cdot\nabla\boldsymbol{b} + \boldsymbol{b}\cdot\nabla\boldsymbol{B}}{\mu_0}$$

$$B^2 = (\boldsymbol{B}+\boldsymbol{b})\cdot(\boldsymbol{B}+\boldsymbol{b}) = B^2 + 2\boldsymbol{B}\cdot\boldsymbol{b} + b^2$$

1st order

To eliminate the pressure term, take curl of the above equation ($\nabla \times \nabla p^T \equiv 0$)

$$\boxed{\gamma\nabla\times\boldsymbol{u} = \nabla\times(\boldsymbol{B}\cdot\nabla\boldsymbol{b} + \boldsymbol{b}\cdot\nabla\boldsymbol{B})}$$

$$\boxed{\text{Let } \frac{\boldsymbol{B}}{\sqrt{\mu_0\rho}} \to \boldsymbol{B}}$$

$$\boxed{\gamma \nabla \times \boldsymbol{u} = \nabla \times (\boldsymbol{B} \cdot \nabla \boldsymbol{b} + \boldsymbol{b} \cdot \nabla \boldsymbol{B})}$$

$$\boldsymbol{B} \cdot \nabla \boldsymbol{b} = ikB_x \boldsymbol{b} \qquad \boldsymbol{b} \cdot \nabla \boldsymbol{B} = b_y \frac{\partial \boldsymbol{B}}{\partial y}$$

Take the z-component of the equation:

$$\nabla \cdot \boldsymbol{b} = ikb_x + b_y' = 0 \longrightarrow b_x = \frac{i}{k} b_y'$$

$$\nabla \cdot \boldsymbol{u} = iku_x + u_y' = 0 \longrightarrow u_x = \frac{i}{k} u_y'$$

$$\boxed{f' = \frac{\partial f}{\partial y}}$$

$$\boxed{\gamma\left(u_y'' - k^2 u_y\right) = ik\left[B_x\left(b_y'' - k^2 b_y\right) - B_x'' b_y\right]}$$

This is an equation of u_y and b_y only

Faraday's equation

$$\frac{\partial \boldsymbol{B}}{\partial t} = \nabla \times (\boldsymbol{V} \times \boldsymbol{B}) + \eta \nabla^2 \boldsymbol{B}$$

$$\big\downarrow$$

$$\boxed{\gamma \boldsymbol{b} = \nabla \times (\boldsymbol{u} \times \boldsymbol{B}) + \eta \nabla^2 \boldsymbol{b}}$$

$$\nabla \times (\boldsymbol{u} \times \boldsymbol{B}) = \boldsymbol{B} \cdot \nabla \boldsymbol{u} - \boldsymbol{u} \cdot \nabla \boldsymbol{B} \qquad (\nabla \cdot \boldsymbol{u} = \nabla \cdot \boldsymbol{B} = 0)$$

$$\boldsymbol{B} \cdot \nabla \boldsymbol{u} = ikB_x \boldsymbol{u} \qquad \boldsymbol{u} \cdot \nabla \boldsymbol{B} = u_y \frac{\partial \boldsymbol{B}}{\partial y}$$

$$\nabla^2 \boldsymbol{b} = \left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}\right) \boldsymbol{b} = \left(-k^2 + \frac{\partial^2}{\partial y^2}\right) \boldsymbol{b}$$

Take the y-component of the equation:

$$\boxed{\gamma b_y = ikB_x u_y + \eta(b_y'' - k^2 b_y)}$$

This is an equation of u_y and b_y only

$$\gamma\left(u_y'' - k^2 u_y\right) = ik\left[B_x\left(b_y'' - k^2 b_y\right) - B_x'' b_y\right]$$

$$\gamma b_y = ikB_x u_y + \eta\left(b_y'' - k^2 b_y\right)$$

Basic equation set for following analysis

For convenience, use u, b, B to replace u_y, b_y, B_x

- B_z does not enter the equation set: it does not affect the growth rate of 1D tearing at all!
  - But it affects the functions $b_z(y)$ and $u_z(y)$. You can derive the equations for them as an exercise.
  - If $\boldsymbol{k} = k_x\,\hat{e}_x + k_z\,\hat{e}_z$, B_z will influence the growth rage (only slightly), see (Shi et al., 2020)
- The equation seems to be *complex* (both real and imaginary terms), but it's **NOT**!
  - Do the transform: $ib \to b$ the equation set will become the following:

purely real

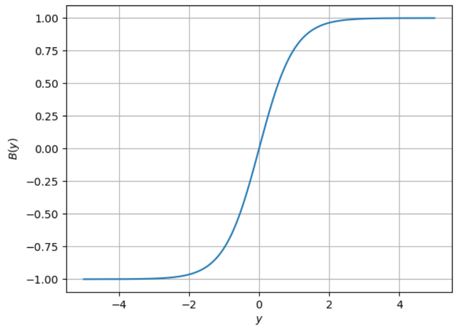
$$\gamma\left(u'' - k^2 u\right) = k\left[B\left(b'' - k^2 b\right) - B'' b\right]$$

$$\gamma b = -kBu + \eta\left(b'' - k^2 b\right)$$

There is no oscillation: tearing mode is purely growing/decaying mode

Parity of u and b

$B = \tanh(y)$



$$\boxed{\begin{aligned}\gamma(u'' - k^2u) &= k[B(b'' - k^2b) - B''b] \\ \gamma b &= -kBu + \eta(b'' - k^2b)\end{aligned}}$$

$B(y)$ is odd, $B'(y)$ is even, $B''(y)$ is odd

Note: odd function × odd function → even function
odd function × even function → odd function

Theoretically, there are two possibilities:

(1) u is even, b is odd
This is not desirable: $b_y(0) \equiv 0$, no reconnection, wavy current sheet

(2) u is odd, b is even
$u_y(0) \equiv 0, b_y(0) \neq 0$. This is reconnection.

The Boundary Layer Problem

Consider $\eta \to 0$

$$\gamma(u'' - k^2 u) = k[B(b'' - k^2 b) - B'' b]$$
$$\gamma b = -kBu + \boxed{\eta(b'' - k^2 b)}$$

The highest-order derivative term is multiplied by a very small coefficient:

boundary layer problem

- The η term is important only in a very thin layer $|y| \lesssim \delta$.
- δ is an unknown quantity depending on η & k

Generally, to solve the boundary layer problem:

- We first consider the **outer region** where $|y| \gg \delta$.
- Then we consider the **inner region** where $|y| \lesssim \delta$.
- Then we match the two solutions at $|y| \sim \delta$ (as the solution must converge here)

$$\boxed{\begin{aligned}\gamma(u'' - k^2 u) &= k[B(b'' - k^2 b) - B'' b] \\ \gamma b &= -kBu + \eta(b'' - k^2 b)\end{aligned}}$$

Outer region $|y| \gg \delta$

Here, the resistive term $(\eta(b'' - k^2 b))$ is negligible

The 2nd equation gives

$$u = -\frac{\gamma}{kB} b$$

Then plug it into the first equation, we get an equation

$$b'' - k^2 b - \frac{B''}{B} b + O(\gamma^2) = 0$$

$O(\gamma^2)$ are terms proportional to γ^2 which can be neglected in the limit $\eta \to 0$, because γ should naturally be zero if there is no resistivity (tearing mode is resistive instability!)

So, we are left with the equation (which is just the r.h.s. of the first equation):

$$\boxed{b'' - k^2 b - \frac{B''}{B} b = 0}$$

$$\boxed{b'' - k^2 b - \frac{B''}{B} b = 0}$$

Consider the Harris-type current sheet

$$B(y) = B_0 \tanh\left(\frac{y}{a}\right)$$

a: current sheet thickness

Note: $a \neq \delta$!

$$\frac{b''}{b} = k^2 - \frac{2}{a^2}\operatorname{sech}^2\left(\frac{y}{a}\right)$$

Fortunately, this equation has analytic solution:

$$b = \begin{cases} e^{-ky}\left(1 + \dfrac{1}{ka}\tanh\left(\dfrac{y}{a}\right)\right), & y \geq 0 \\ e^{ky}\left(1 - \dfrac{1}{ka}\tanh\left(\dfrac{y}{a}\right)\right), & y \leq 0 \end{cases}$$

Note: we have imposed the requirement:

$$b(y = \pm\infty) = 0$$

Otherwise, both the two branches are exact solutions on $y \in (-\infty, +\infty)$

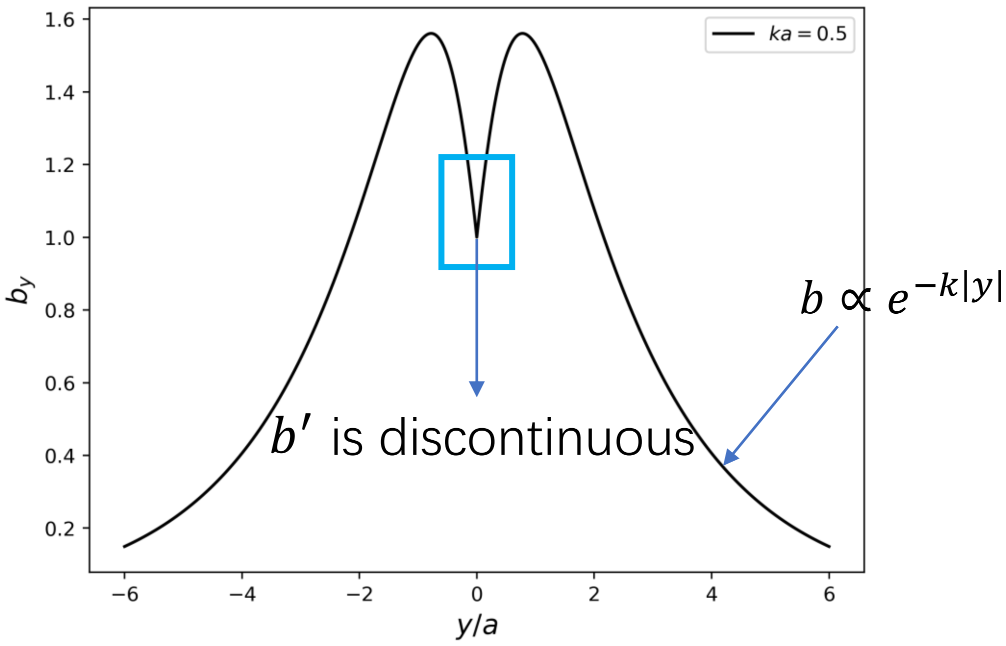
$$b = \begin{cases} e^{-ky}\left(1 + \frac{1}{ka}\tanh\left(\frac{y}{a}\right)\right), y \geq 0 \\ e^{ky}\left(1 - \frac{1}{ka}\tanh\left(\frac{y}{a}\right)\right), y \leq 0 \end{cases}$$

Far from the center of the current sheet, both b and u decay exponentially ($\propto \exp(-ky)$)

As $y \to 0$, b' is discontinuous, i.e.

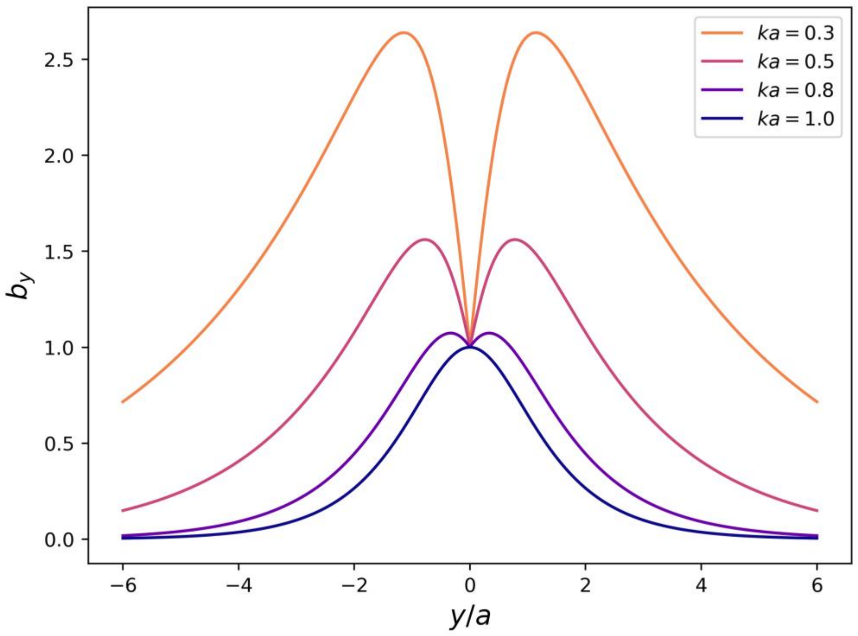
$$b'(y = 0^+) \neq b'(y = 0^-)$$

as $b'(y = 0^+) = -b'(y = 0^-)$. Thus, to ensure $b''(y = 0)$ is finite, we need $b'(y = 0^+) = b'(y = 0^-) = 0$



Let's define a useful quantity derived from the ***outer solution***:

$$\Delta = a\frac{b'(0^+) - b'(0^-)}{b(0)} = \frac{2(1 - k^2a^2)}{ka}$$



Then, we need to analyze the inner layer $|y| \lesssim \delta$

$$\boxed{\begin{aligned}\gamma(u'' - k^2 u) &= k[B(b'' - k^2 b) - B'' b]\\ \gamma b &= -kBu + \eta(b'' - k^2 b)\end{aligned}}$$

Inner region $|y| \lesssim \delta$

We are very close to $y = 0$. Let's do some comparisons.

$$B = B_0 \tanh(\frac{y}{a}), \quad B'' = -\frac{2B_0}{a^2} \tanh\left(\frac{y}{a}\right) \mathrm{sech}^2\left(\frac{y}{a}\right)$$

At $y \to 0$, we have: $\frac{B'' b}{B b''} = -2\,\mathrm{sech}^2\left(\frac{y}{a}\right) \frac{b}{a^2 b''} \sim \boxed{\frac{b}{a^2 b''} \ll 1}$ $\quad b'' \propto (\frac{1}{a\delta}\ or \frac{1}{\delta^2}) \gg \frac{1}{a^2}$

Because the layer is very thin so $b \& u$ change very fast

Similarly, we expect $\frac{k^2 u}{u''} \ll 1, \frac{k^2 b}{b''} \ll 1$

$$\boxed{\gamma(u''}-k^2u)=\boxed{k[B(b''}-k^2b)-B''b]$$
$$\boxed{\gamma b}=\boxed{-kBu}+\boxed{\eta(b''}-k^2b)$$

Inside the inner layer, we have:

$$\boxed{\begin{gathered}\gamma u'' \sim kBb'' \\ \gamma b \sim -kBu \sim \eta b''\end{gathered}}$$

From now on, we are considering all the fields at a location $y=\varepsilon\delta$ with $\varepsilon \lesssim 1$.
For convenience, we throw away ε and write $y=\delta$.

Hint: Taylor expansion: $u(0)=u(\delta)+\cdots$

First, we have $B(\delta)\approx B'(0)\delta$.

Second, as u is an odd function ($u(0)=0$), we have: $u''(\delta)\approx-\frac{u(\delta)}{\delta^2}$.

Last, **estimate of b'' should be made with caution**, as b is an even function.

Taylor expansion:

$$b'(0) = b'(\delta) - \delta \times b''(\delta) + O(\delta^2)$$

Note that $b'(0) = 0$, we have:

$$b''(\delta) \sim \frac{b'(\delta)}{\delta}$$

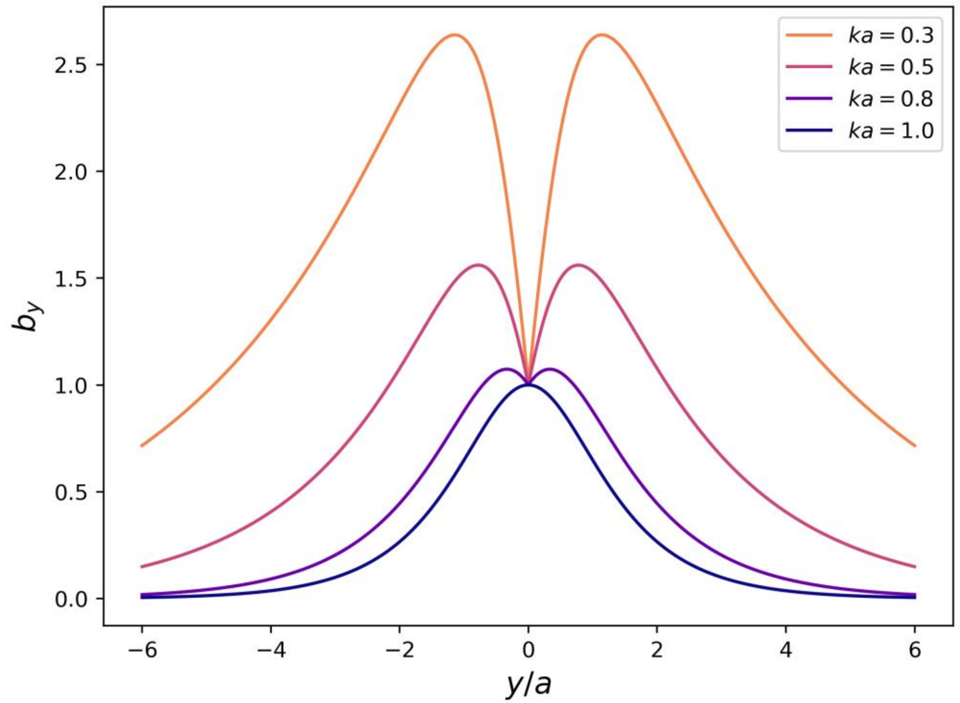
Recall the definition: $\Delta = a\frac{b'(0^+)-b'(0^-)}{b(0)} = \frac{2(1-k^2a^2)}{ka}$.

Note that $0^\pm$ refers to $\pm\delta$ here because this is from the outer solution.

We can write $b'(\delta) \approx \Delta b(0)/a$ and then:

$$b''(\delta) \sim \frac{\Delta b(0)}{a\delta} \approx \frac{\Delta b(\delta)}{a\delta}$$

as $b(\delta) = b(0) + O(\delta^2) \approx b(0)$



ψ is the magnetic flux function such that $\nabla \times (\psi \hat{e}_z) = \boldsymbol{b}$ and thus $\psi = \frac{ib_y}{k} \propto b$

Note: for now we are assuming $ka \gg \frac{\delta}{a}$, i.e., the wave length is not too long (or k is not too small) so that Δ is a finite value and $b'' \propto \delta^{-1}$. This is the traditionally-called ***constant-ψ*** regime.

Because ψ, or b, is smooth

However, as ka becomes smaller, b becomes steeper (see the figure above). When $ka \lesssim \frac{\delta}{a} \ll 1$, we find that $\Delta \approx \frac{2a}{\delta}$ and thus $b'' \propto \delta^{-2}$, similar to u. This is the so-called ***non-constant-ψ*** regime.

Because ψ, or b, is very steep

constant-ψ regime $(ka \gg \frac{\delta}{a})$

Balance relations: $\boxed{\begin{array}{c} \gamma u'' \sim kBb'' \\ \gamma b \sim -kBu \sim \eta b'' \end{array}}$

Now we have: $u'' \sim -u/\delta^2$, $B \sim B'\delta$, $b'' \sim \Delta b/(a\delta)$

From $\gamma b \sim \eta b''$ we get (use the above estimate of b''):

$$\boxed{\gamma \sim \frac{\eta\Delta}{a\delta}}$$

Then, from $\gamma u'' \sim kBb''$ we get

$$\boxed{\gamma u \sim -kB'\Delta\delta^2 b/a}$$

In addition, we have the relation $\gamma b \sim -kBu$, which gives

$$\boxed{\gamma b \sim -kB'\delta u}$$

We see Δ must be positive if we want γ to be real (instability)

$$\boxed{\gamma^2 \sim \frac{(kB')^2\delta^3}{a}\Delta}$$

Then we can combine the three relations to eliminate u, b and δ and get:

$$\gamma^5 \sim (kB')^2\Delta^4\eta^3/a^4$$

$$\gamma^5 \sim \frac{(kB')^2 \Delta^4 \eta^3}{a^4}$$

$$\boxed{\gamma\tau_a \sim \left(\frac{aB'}{\bar{B}}\right)^{\frac{2}{5}} (1 - k^2a^2)^{\frac{4}{5}} (ka)^{-\frac{2}{5}} S^{-\frac{3}{5}}}$$

$$\Delta = \frac{2(1 - k^2a^2)}{ka}$$

$$S = \frac{a\bar{B}}{\eta}$$

$\bar{B}$ is the upstream Alfvén speed

$\tau_a = a/\bar{B}$: Alfvén crossing time

As we see previously, Δ should be positive in order to have a growing mode, i.e., the wave length must be longer than a

$$ka < 1$$

Then one can easily show that $(1 - k^2a^2)^{\frac{4}{5}}(ka)^{-\frac{2}{5}}$ is a **decreasing** function of k for $0 < ka < 1$.

Especially, when $\delta/a \ll ka \ll 1$, we have $\gamma \propto (ka)^{-\frac{2}{5}} S^{-\frac{3}{5}}$

non-constant-ψ regime $(ka \lesssim \frac{\delta}{a})$

Balance relations:
$$\boxed{\begin{array}{c} \gamma u'' \sim kBb'' \\ \gamma b \sim -kBu \sim \eta b'' \end{array}}$$

Now we have: $u'' \sim -u/\delta^2$, $B \sim B'\delta$, $b'' \sim b/\delta^2$

From $\gamma b \sim \eta b''$ we get (use the above estimate of b''):
$$\boxed{\gamma \sim \frac{\eta}{\delta^2}}$$

Then, from $\gamma u'' \sim kBb''$ we get
$$\boxed{\gamma u \sim -kB'\delta b}$$

In addition, we have the relation $\gamma b \sim -kBu$, which gives
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$$\boxed{\gamma\tau_a \sim \left(\frac{aB'}{\bar{B}}\right)^{\frac{2}{3}} (ka)^{\frac{2}{3}} S^{-1/3}}$$

$$S = \frac{a\bar{B}}{\eta}$$

$\bar{B}$ is the upstream Alfvén speed

$\tau_a = a/\bar{B}$: Alfvén crossing time

In contrast to the constant-ψ regime, now γ is an **increasing** function of k

	constant-ψ	non-constant-ψ
k range	$\delta/a \ll ka < 1$	$ka \lesssim \delta/a$
$\gamma \sim$	$(1-k^2a^2)^{\frac{4}{5}}(ka)^{-\frac{2}{5}}S^{-\frac{3}{5}}$	$(ka)^{\frac{2}{3}}S^{-1/3}$
γ and k	Decreasing with k	Increasing with k

Apparently, there is a critical point around $ka \sim \delta/a$ that separates the two regimes. And this critical point corresponds to the **peak** of the $\gamma(k)$ curve.

Around the critical point (i.e. $ka \sim \delta/a$), we have

$$\gamma\tau_a \sim (ka)^{-\frac{2}{5}} S^{-\frac{3}{5}}$$

for constant-ψ regime (large-k branch), and

$$\gamma\tau_a \sim (ka)^{\frac{2}{3}} S^{-\frac{1}{3}}$$

for non-constant-ψ regime (small-k branch).

Thus, to match the two solutions, we have:

$$(k_m a)^{-\frac{2}{5}} S^{-\frac{3}{5}} \sim (k_m a)^{\frac{2}{3}} S^{-\frac{1}{3}}$$

which gives

$$k_m a \sim S^{-\frac{1}{4}}$$

k_m is the wavenumber corresponding to the maximum growth rate

$$\boxed{k_m a \sim S^{-\frac{1}{4}}}$$

Then we plug k_m into the expression for γ to get the maximum γ:

$$\boxed{\gamma_m \tau_a \sim S^{-\frac{1}{2}}}$$

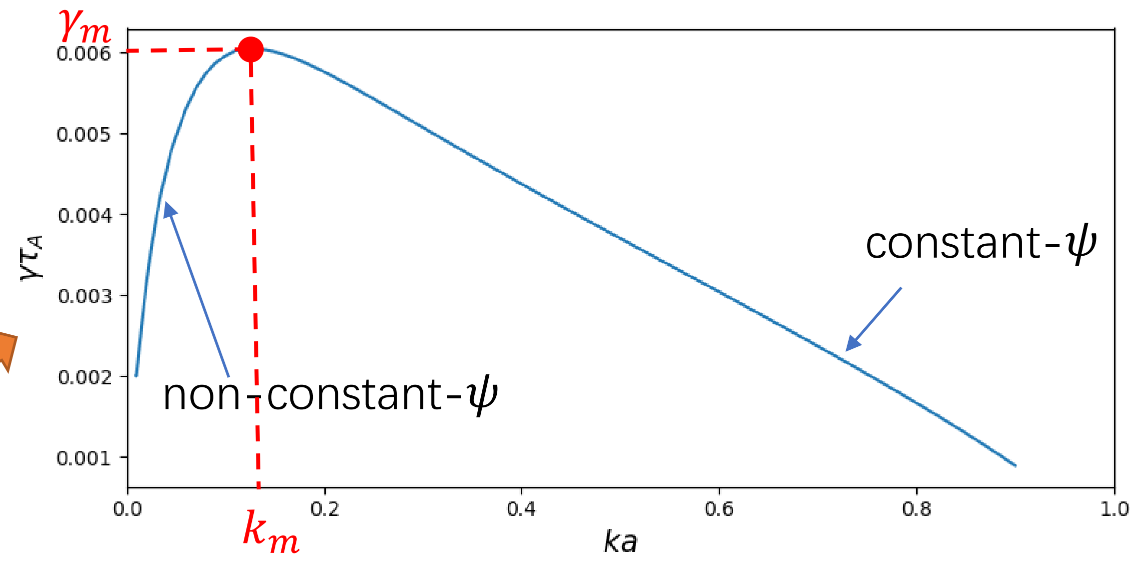
One can test using either $\gamma \sim \frac{\eta\Delta}{a\delta}$ or $\gamma \sim \frac{\eta}{\delta^2}$ (the balance relations $\gamma b \sim \eta b''$ for the two regimes) that at this point, there is

$$\frac{\delta_m}{a} \sim S^{-1/4}$$

consistent with the assumption $k_m a \sim \delta_m / a$

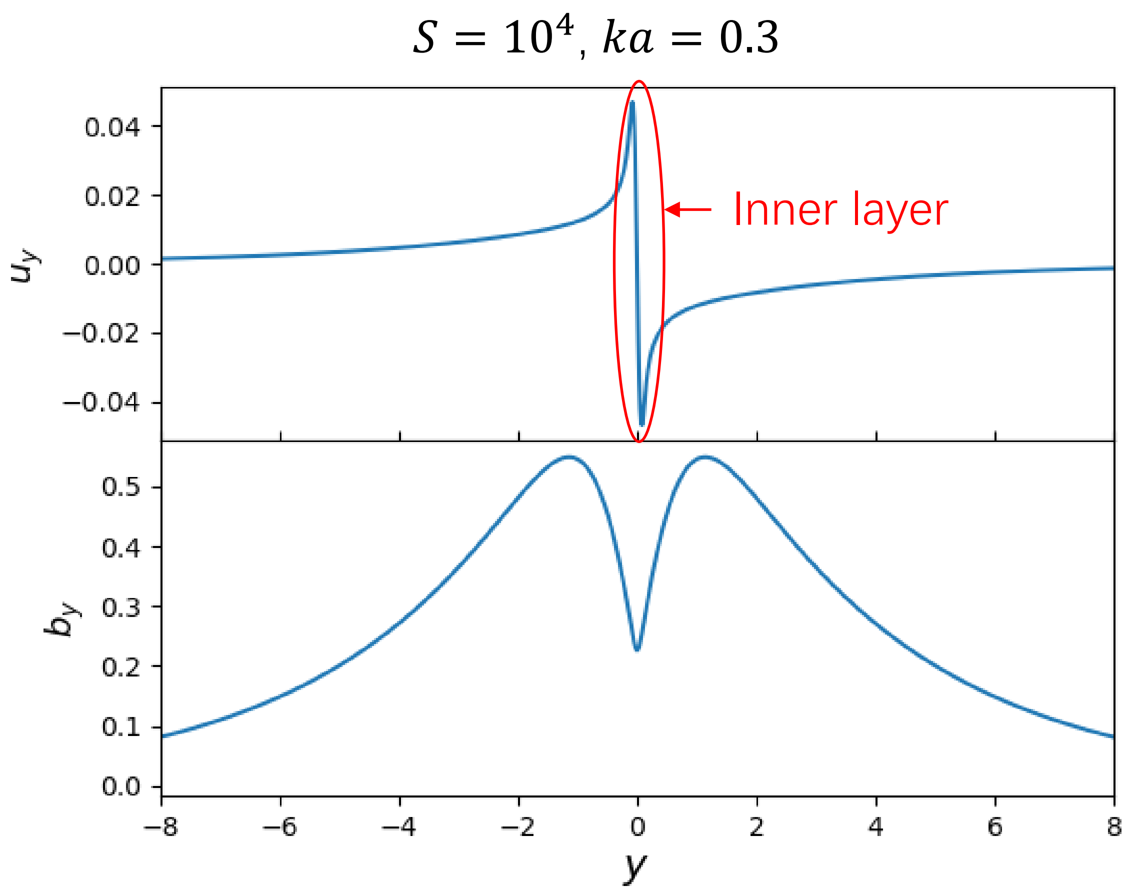
To conclude this part

- The linear tearing instability requires $ka < 1$
- The $\gamma(k)$ curve consists of two parts
  - The large-k, or constant-ψ, regime where γ decreases with k
  - The small-k, or non-constant-ψ, regime where γ increases with k
- Most importantly, the intersection of the two regimes is the fastest-growing mode, which has the following scaling relations:
  - $\delta_m/a \sim S^{-\frac{1}{4}}$
  - $\gamma_m\tau_a \sim S^{-\frac{1}{2}}$
  - $k_m a \sim S^{-\frac{1}{4}}$



Numerically solved $\gamma(k)$ curve for $S = 10^4$

Numerically solved eigen-functions



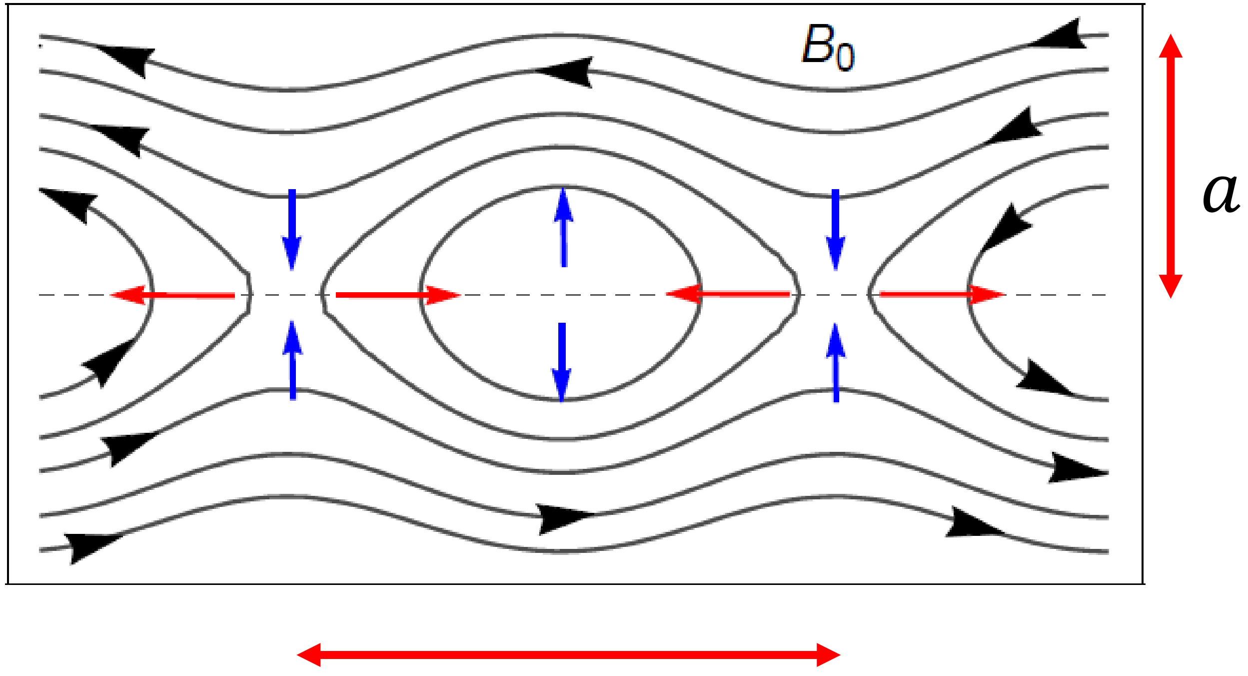
Example of a numerically-solved $u\&b$

$$\gamma(u'' - k^2 u) = k[B(b'' - k^2 b) - B'' b]$$
$$\gamma b = -kBu + \eta(b'' - k^2 b)$$

The equation set that we are solving is a ***boundary-value problem***, i.e. an ODE set with boundary condition $(u, b)|_{y=\pm\infty} = 0$ and an undetermined ***eigenvalue*** γ. This kind of problem can be solved numerically with existing packages (e.g. SciPy solve_bvp)

In practice, we impose the boundary condition $(u, b) \propto e^{-k|y|}$ at two boundaries far from the center of the current sheet (because we cannot impose boundary conditions at infinity numerically).

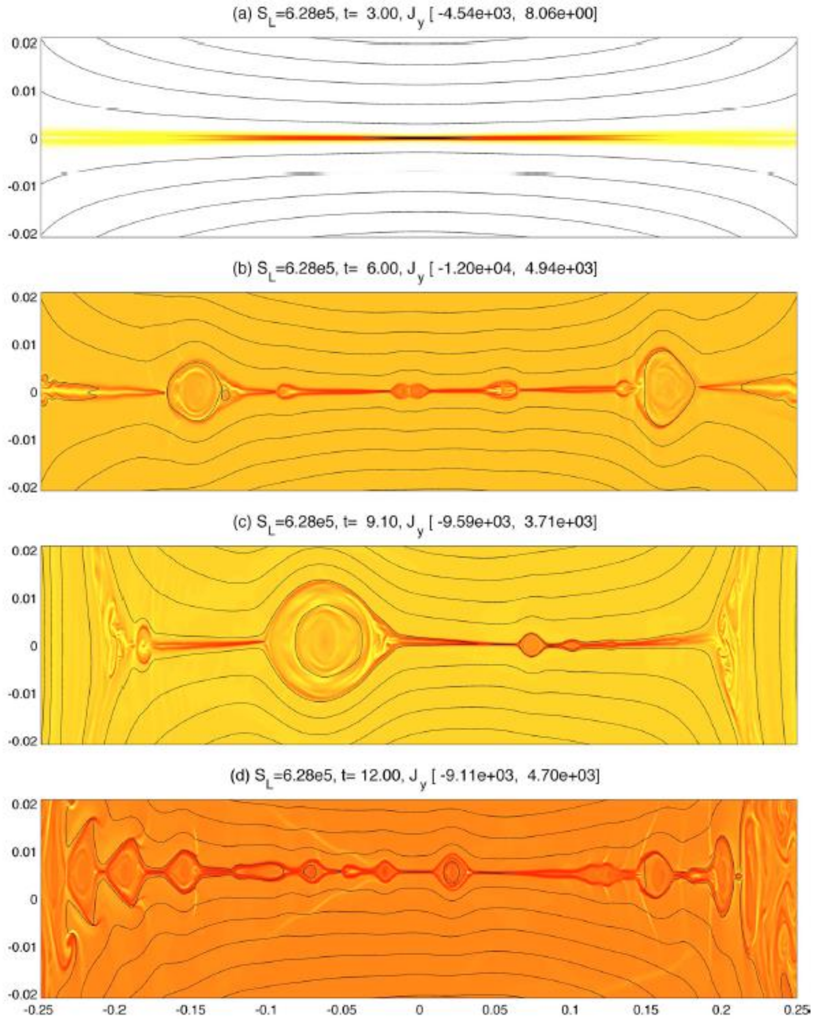
Nonlinear stage of tearing mode -- Plasmoid



$$\lambda_m = \frac{2\pi}{k_m} = 2\pi a S^{1/4}$$

Q: how to estimate the width of the plasmoid?

A: $w \cong 4\sqrt{\psi/\psi_0''}$ (Biksamp 2005 Eq (4.5))



(Bhattacharjee et al, 2009)

The tearing instability seems to be slow…

Even the fastest growing mode has a ***negative*** scaling relation with S: $\gamma_m \tau_a \sim S^{-\frac{1}{2}}$. This means that, the larger S, or the smaller η, leads to a smaller $(\gamma_m \tau_a)$. And $S \to \infty$ gives $(\gamma_m \tau_a) \to 0$.

This is true. But, we should notice that, so far, we are only discussing the ***one-dimensional*** current sheet, i.e. the current sheet is infinite along x direction, and we normalize everything to the thickness of the current sheet a. However, the current sheet must have a finite length L, for example, the Sweet-Parker current sheet.

Aspect ratio and the growth rate

Recall: in Sweet-Parker current sheet model, the Lundquist number is defined as

$$S_L = \frac{LV_A}{\eta}$$

instead of

$$S = \frac{aV_A}{\eta}$$

For a 2D current sheet, as its thickness is usually dynamically evolving, we should use its length to measure the time scales (convection, diffusion etc.)

Let's renormalize everything by L instead of a. We have

$$\tau_L = \frac{L}{V_A} = \tau_a \times \frac{L}{a}$$

$$S_L = S_a \times \frac{L}{a}$$

Exercise: show the wavenumber of the maximum growing mode is

$$k_m L \sim S_L^{\frac{3}{8}}$$

for Sweet-Parker CS

$$\gamma_m \tau_a \sim S^{-\frac{1}{2}}$$

$$\Downarrow$$

$$\gamma_m \tau_L \times \frac{a}{L} \sim S_L^{-\frac{1}{2}} \times \left(\frac{a}{L}\right)^{-\frac{1}{2}}$$

$$\Downarrow$$

$$\gamma_m \tau_L \sim S_L^{-\frac{1}{2}} \boxed{\times \left(\frac{a}{L}\right)^{-\frac{3}{2}}}$$

Aspect ratio is important!

Recall: Sweet-Parker current sheet has aspect ratio

$$\frac{a}{L} \sim S_L^{-\frac{1}{2}}$$

(Loureiro et al., 2007)
(Tajima & Shibata, 2003)

This gives

$$\boxed{(\gamma_m \tau_L)_{SP} \sim S_L^{\frac{1}{4}}}$$

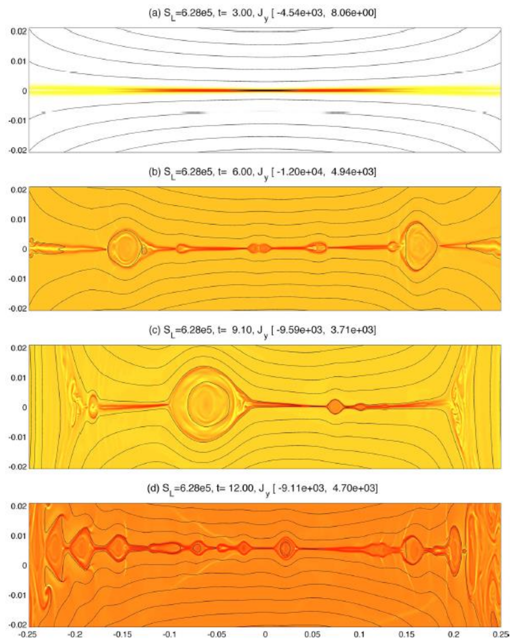
Positive scaling with S_L:
$\gamma_m \tau_L \to \infty$ as $S_L \to \infty$

For Sweet-Parker current sheet ($\frac{a}{L} \sim S_L^{-1/2}$), the linear growth rate of tearing instability is fast:

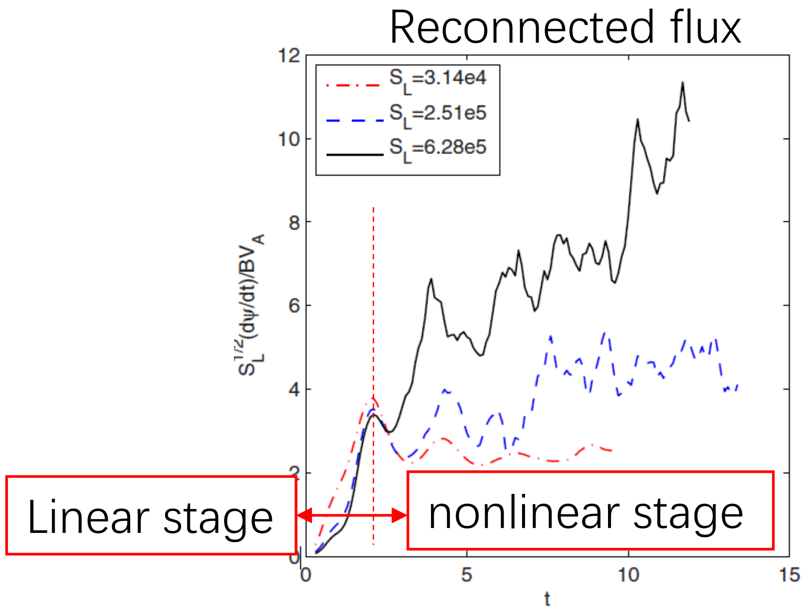
$$(\gamma_m \tau_L)_{SP} \sim S_L^{\frac{1}{4}}$$

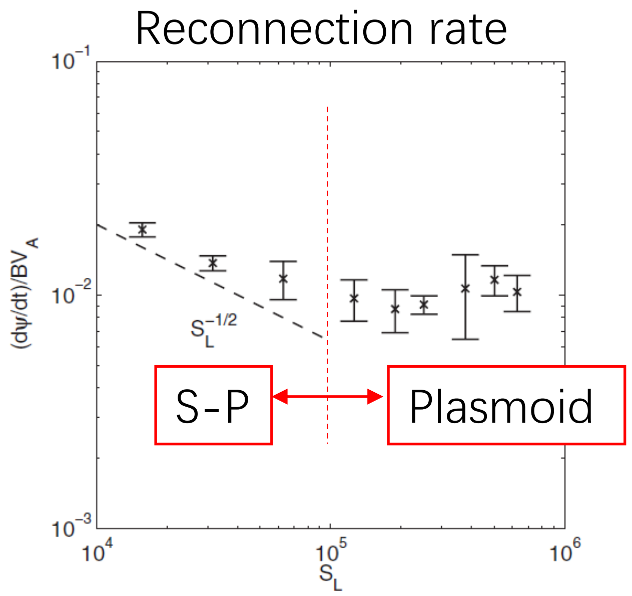
which means $\gamma_m \tau_L \to \infty$ as $S_L \to \infty$.

In practice, the tearing mode cannot grow indefinitely because the growth will saturate when the amplitude is too large, i.e. the mode enters nonlinear stage.



(Bhattacharjee et al, 2009)





“ideal” tearing mode

$$\gamma_m \tau_L \sim S_L^{-\frac{1}{2}} \times \left(\frac{a}{L}\right)^{-\frac{3}{2}}$$

If we consider a “collapsing” current sheet, i.e. a thinning current sheet, whose aspect ratio a/L is decreasing. Write

$$\frac{a}{L} \sim S_L^{-\alpha}$$

we get

$$\gamma_m \tau_L \sim S_L^{-(1-3\alpha)/2}$$

Consider the case when $S_L \to \infty$:

Thin current sheet is extremely unstable

If $\alpha > \frac{1}{3}$ (current sheet too thin): positive scaling with S_L, the growth rate diverges to infinity.

If $\alpha < \frac{1}{3}$ (current sheet too thick): negative scaling with S_L, the growth rate goes to zero.

If $\alpha = \frac{1}{3}$: finite growth rate $\gamma_m \tau_L \sim \boxed{O(1)}$

independent of S_L

Thick current sheet is extremely stable

Current sheet is mildly unstable

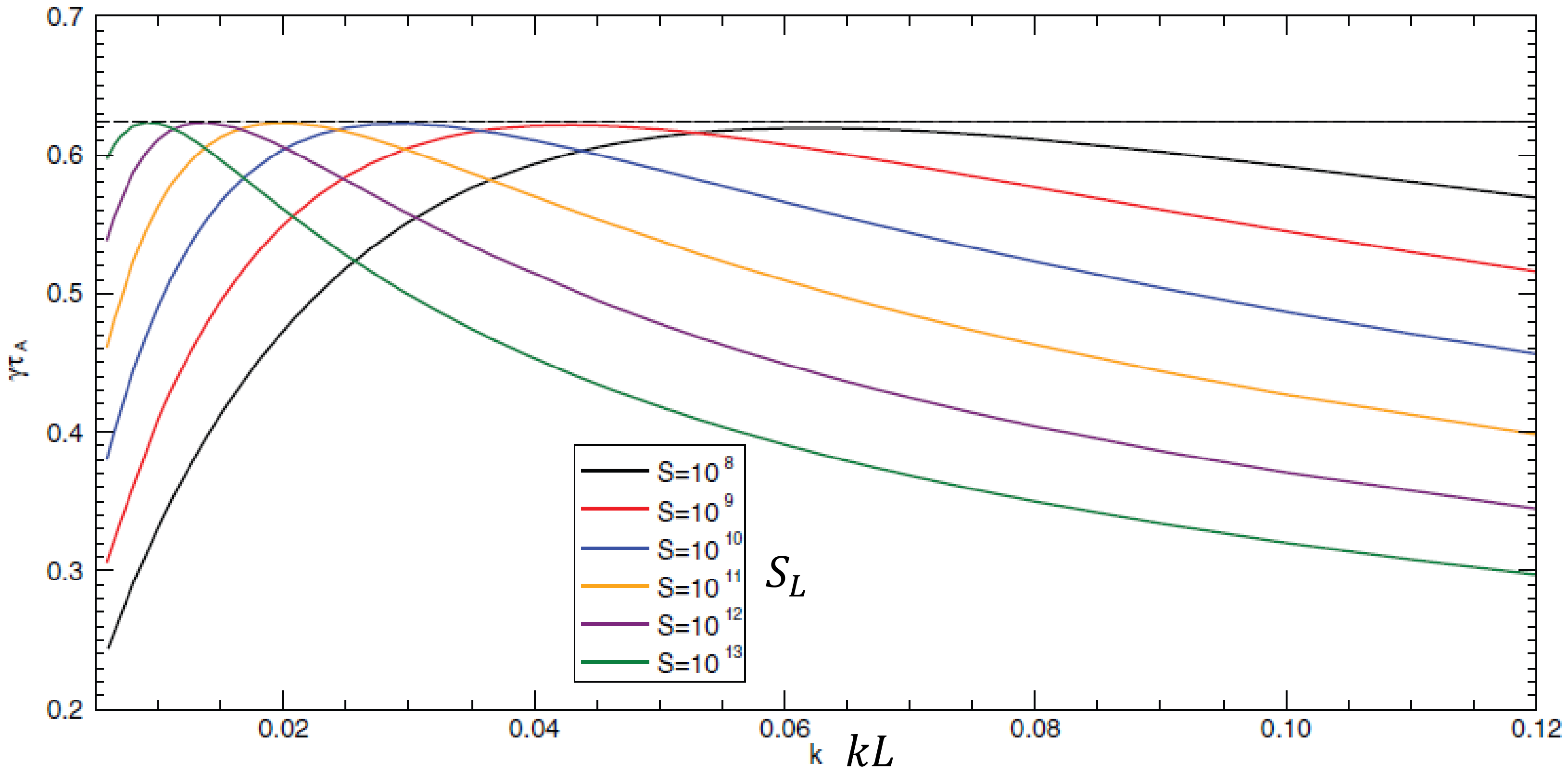
“ideal” tearing mode

(Pucci et al, 2014)

$$\gamma_m \tau_L \sim S_L^{-(1-3\alpha)/2}$$

If $\alpha = \frac{1}{3}$: finite growth rate $\gamma_m \tau_L \sim O(1)$, independent of S_L

"ideal" tearing mode



(Pucci et al, 2014)

Recursive reconnection process

An initially thick current sheet with $\frac{a}{L} > S_L^{-1/3}$: this current sheet is very stable.

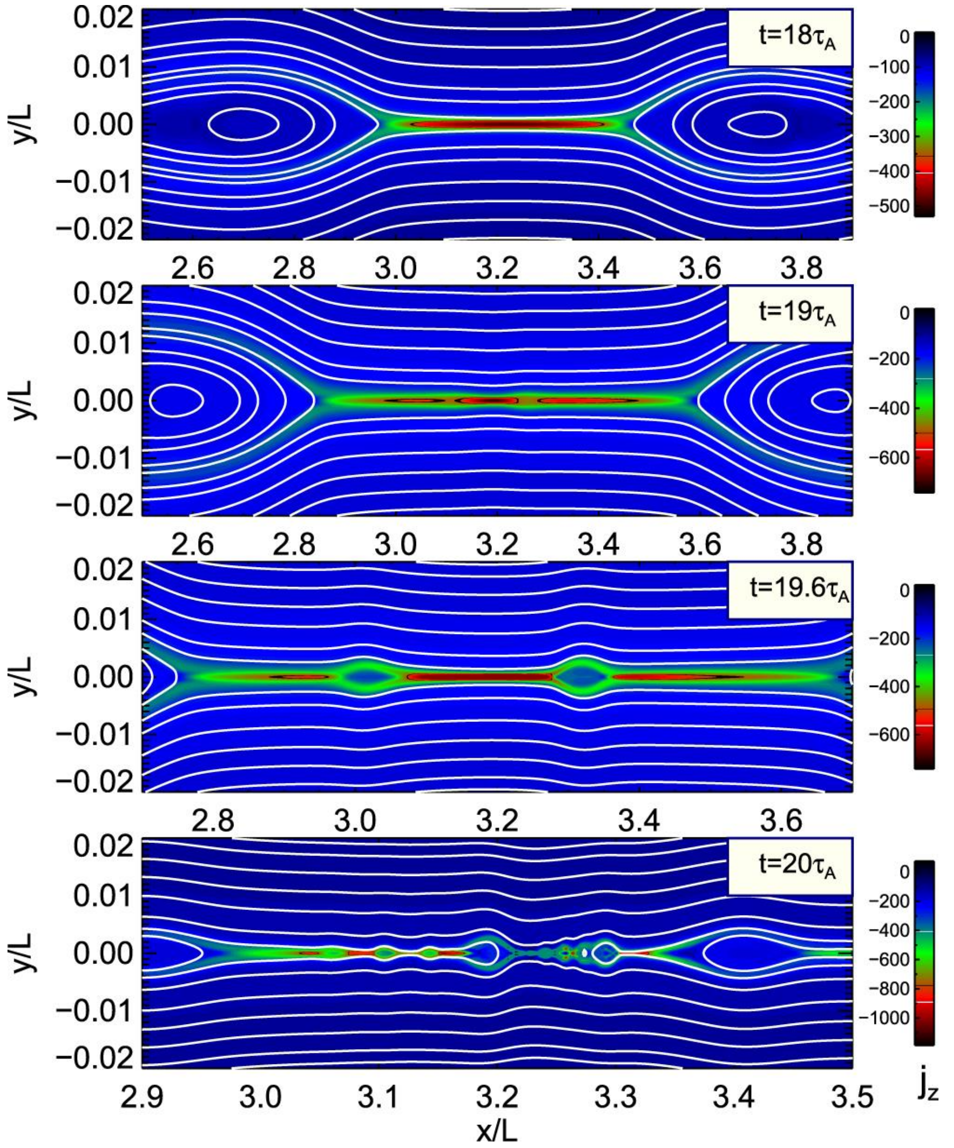
↓

Due to some mechanism, the current sheet thins, and eventually thins to $\frac{a}{L} \sim S_L^{-1/3}$.

↓

Then the tearing instability grows, leading to formation of plasmoid chain

↓



(Tenerani et al, 2015b)

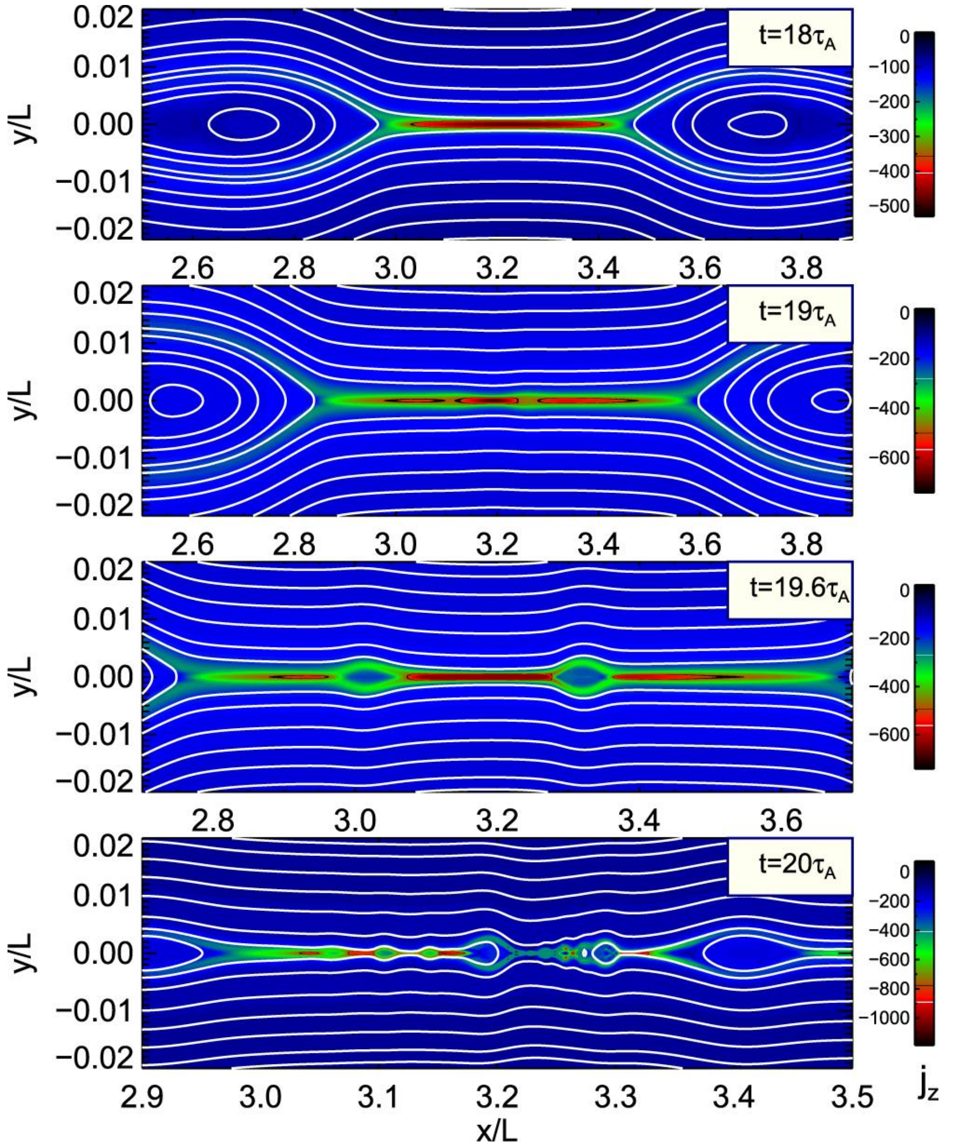
The X-points between plasmoids dynamically lengthen (nonlinear process) and become the secondary current sheets

The secondary current sheets approach the critical aspect ratio ($\frac{a}{L} \sim S_L^{-1/3}$)

Secondary current sheets become tearing unstable and new plasmoids form

Generates a power spectrum: $P(B_y) \propto k_x^{-\frac{4}{5}}$

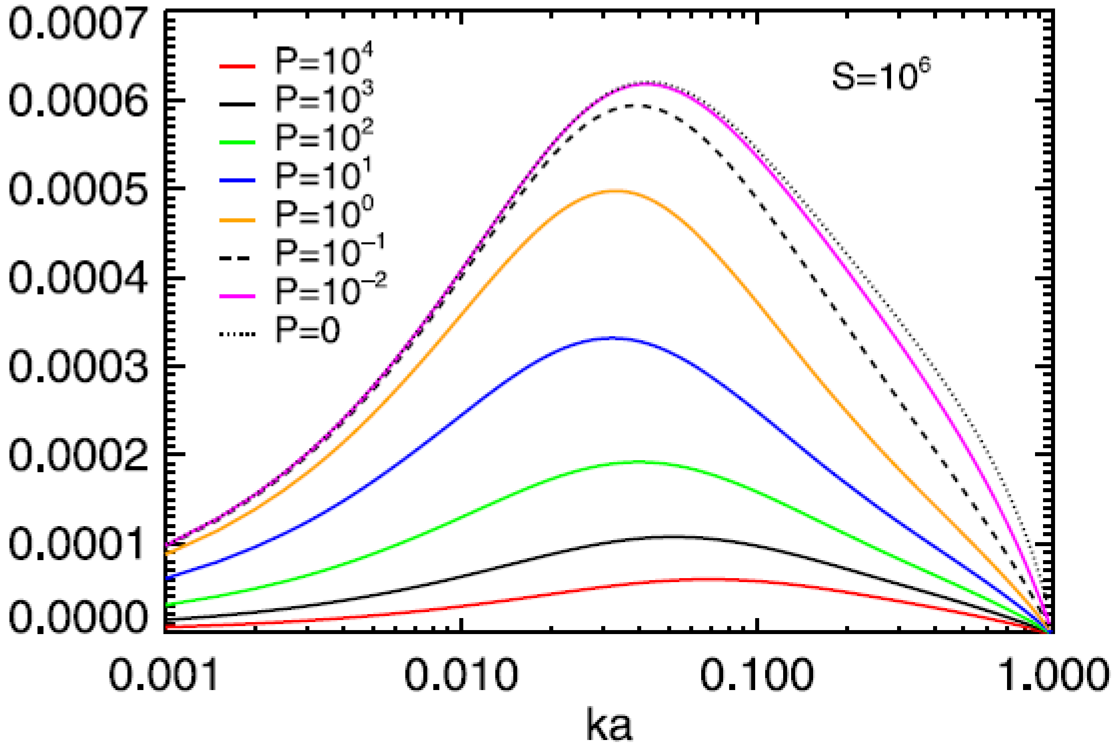
(Tenerani and Velli, 2019)



(Tenerani et al, 2015b)

Other effects – viscosity

$P = \nu/\eta$ (ν: viscosity)



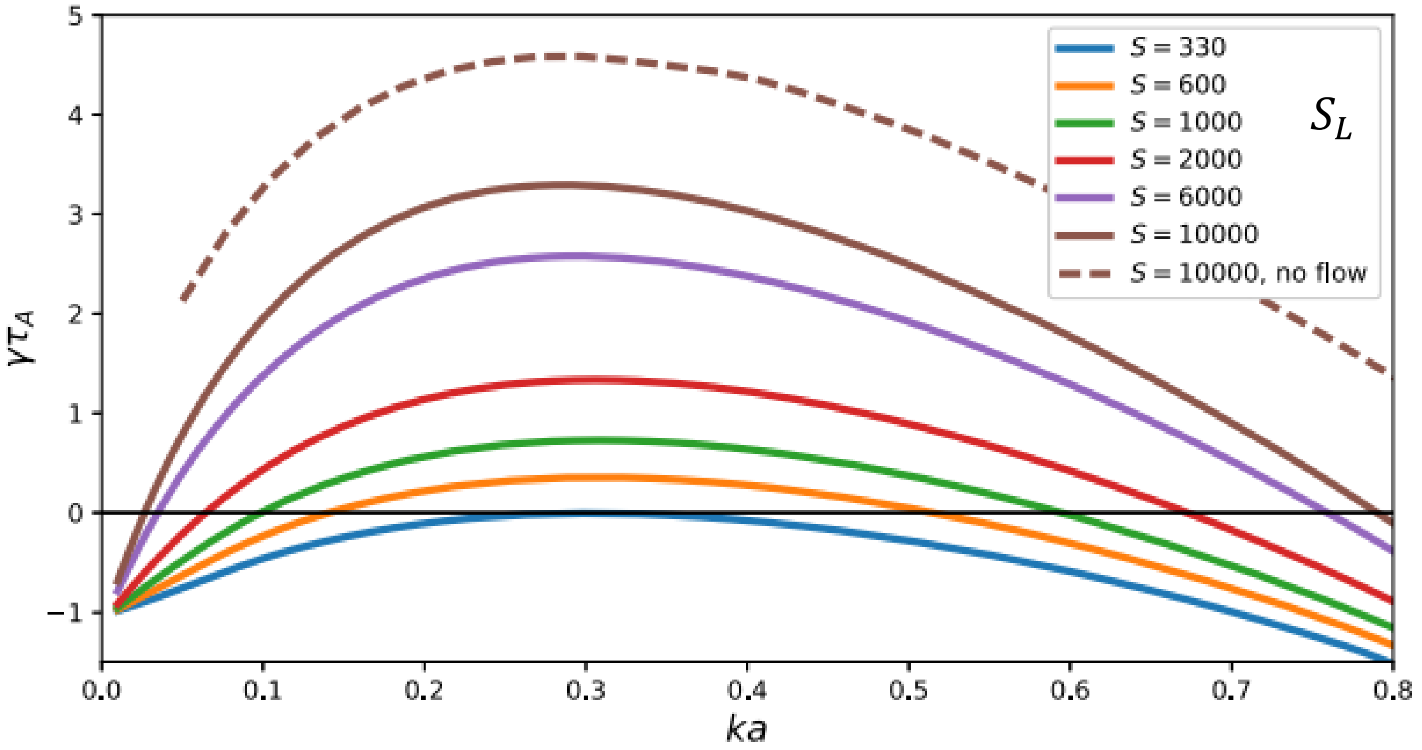
(Tenerani et al, 2015a)

$$\gamma_m \tau_a \sim S^{-\frac{1}{2}} P^{-\frac{1}{4}}$$

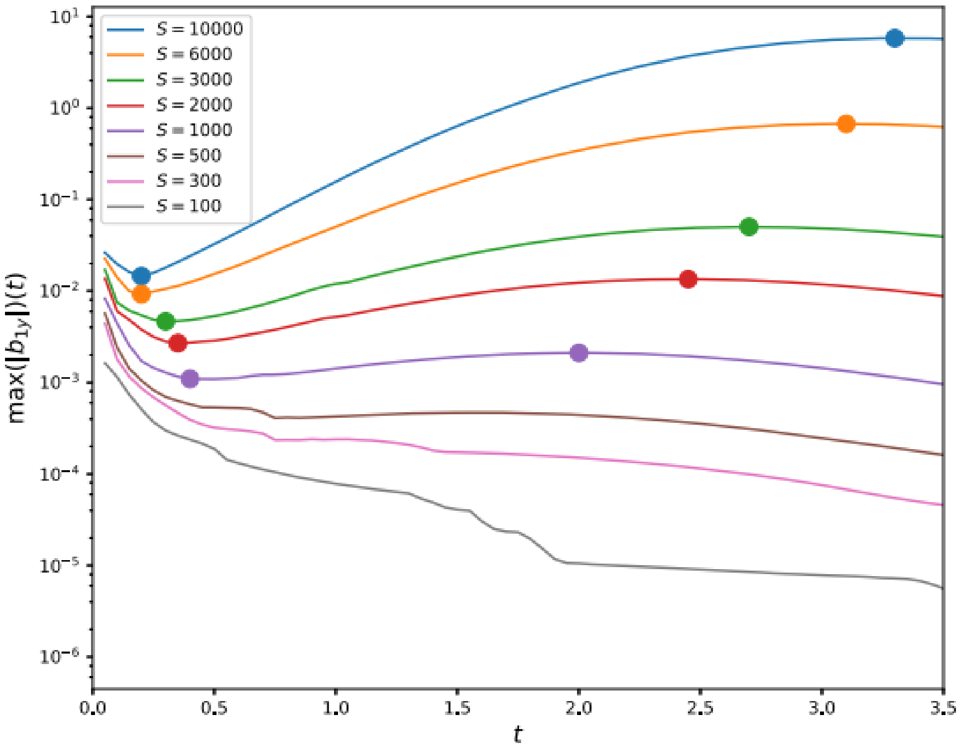
See also (Loureiro et al, 2013)

Viscosity suppresses the tearing mode

Other effects – background flow



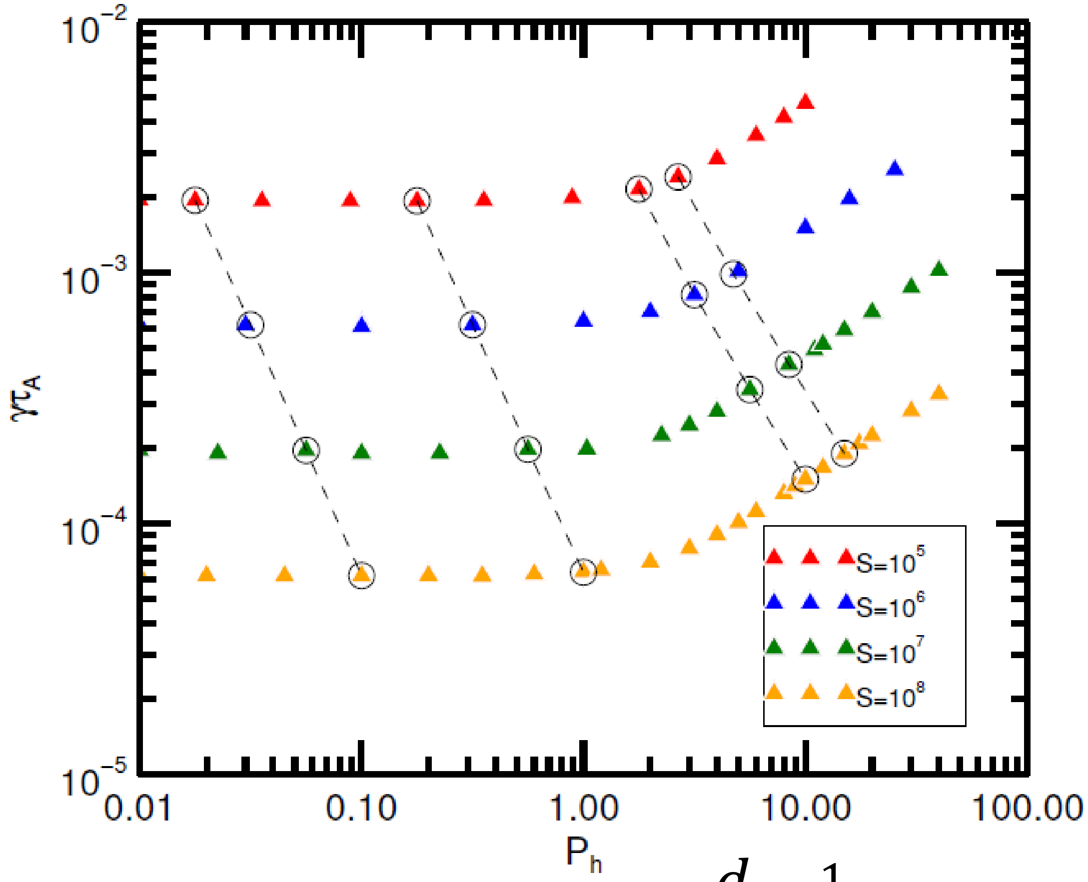
(Shi et al, 2018)



A Sweet-Parker type background flow (inflow+outflow):

1. Decreases the growth rate
2. Leads to a linear "saturation" due to the stretch of the magnetic island by flows

Other effects – Hall effect



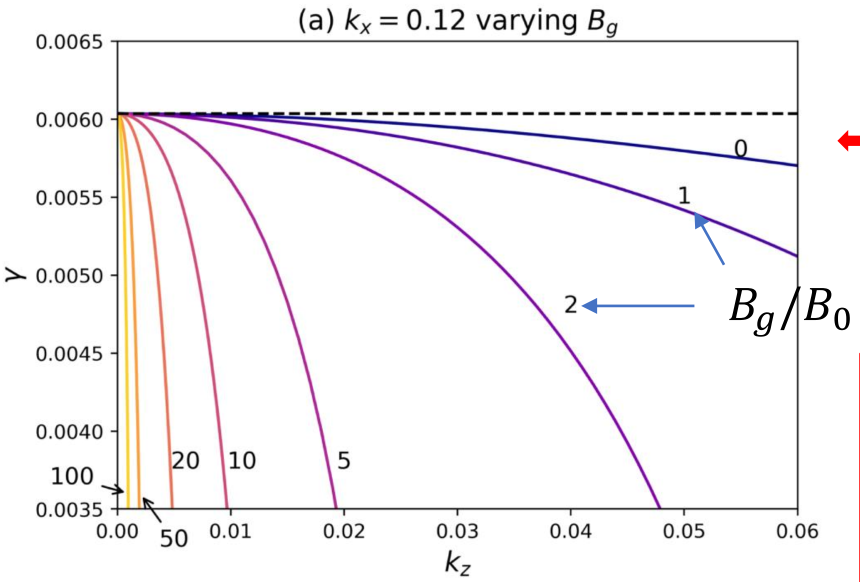
(Pucci et al, 2017)

$$P_h = \frac{d_i}{a} S^{\frac{1}{4}}$$

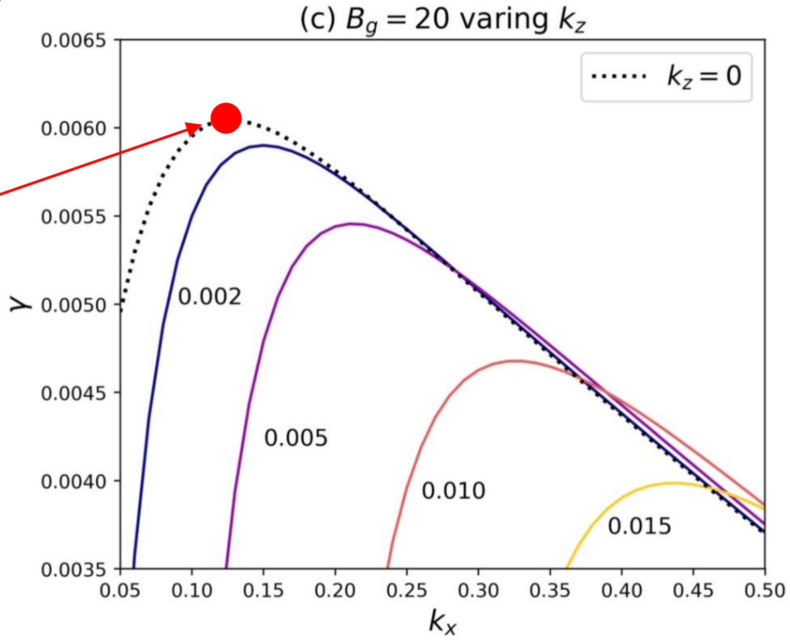
$$\gamma_m \tau_a \sim S^{-\frac{1}{2}}(1 + C P_h^{\zeta})$$

Hall effect increases the linear growth rate of tearing mode.

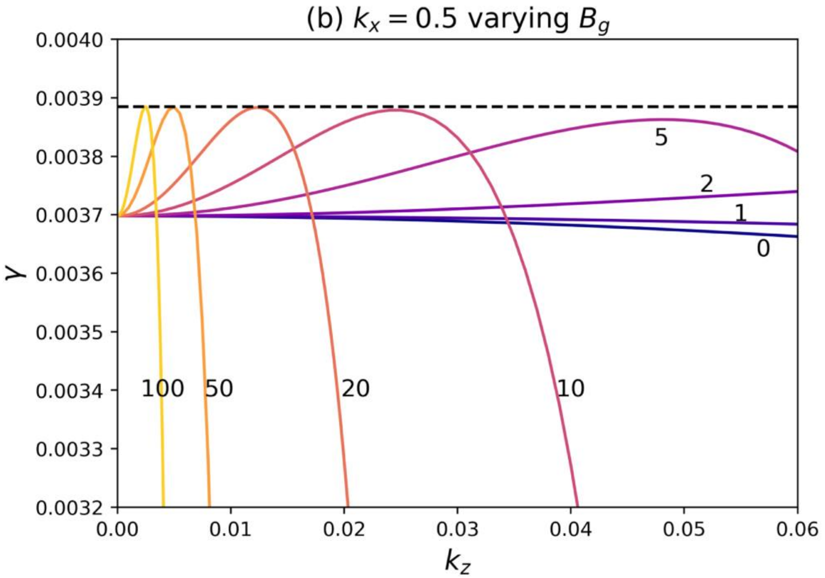
Tearing mode in 3D – guide field



Small k_x regime: γ always decrease with k_z, i.e. a wave vector component along the guide field slows down the growth.



But overall, the fastest growing mode is always the $k_z = 0$ mode, which is not affected by guide field at all.



Large k_x (constant-ψ) regime: guide field can rise γ a little bit with finite k_z.

(Shi et al, 2020)

Summary

- A resistive current sheet is subject to tearing mode instability
- The instability requires **long wavelength**: $ka < 1$
- Linear growth rate $\gamma(k)$ is a non-monotonic function:
  1. For large k, $\frac{d\gamma}{dk} < 0$, for small k, $\frac{d\gamma}{dk} > 0$
  2. The largest growth rate is located at $k_m a \sim S^{-1/4}$ and has a value $\gamma_m \tau_a \sim S^{-1/2}$
- Nonlinearly, the tearing instability leads to the formation of a chain of plasmoids whose length corresponds to k_m
- In a thinning current sheet, the growth rate transits from extremely slow to extremely fast when $\frac{a}{L} \sim S_L^{-1/3}$