MHD equations derived from Vlasov equation

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The Vlasov equation for one species in the plasma is written as

$$\frac{\partial f}{\partial t} + \mathbf{v} \cdot \nabla f + \frac{q}{m}(\mathbf{E} + \mathbf{v} \times \mathbf{B}) \cdot \frac{\partial f}{\partial \mathbf{v}} = (\frac{\partial f}{\partial t})_c \tag{1}$$

where the R.H.S. term is the collision term.

1 Mass conservation equation

Directly integrate Eq (1) over the velocity space $\int d^3\mathbf{v}$ and neglect the collision term, we get

$$\frac{\partial n}{\partial t} + \nabla \cdot (n\mathbf{u}) = 0 \tag{2}$$

where $n = \int f d^3\mathbf{v}$ and $\mathbf{u} = \frac{1}{n}\int \mathbf{v} f d^3\mathbf{v}$ are the number density and bulk velocity which are functions of $\mathbf{x}$ and t only.

2 Momentum conservation equation

Apply the integral $\int \mathbf{v} d^3\mathbf{v}$ to Eq (1), we will get the momentum equation.

2.1

$$\int \mathbf{v}\frac{\partial f}{\partial t} d^3\mathbf{v} = \frac{\partial}{\partial t}(\int \mathbf{v} f d^3\mathbf{v}) = \frac{\partial}{\partial t}(n\mathbf{u}) \tag{3}$$

2.2

$$\begin{aligned}
\int \mathbf{v} \cdot \frac{\partial f}{\partial \mathbf{x}} \mathbf{v} d^3\mathbf{v} &= \frac{\partial}{\partial x_k}[\int v_k v_i f d^3\mathbf{v}] \\
&= \frac{\partial}{\partial x_k}[u_k u_i \int f d^3\mathbf{v} + u_k \int w_i f d^3\mathbf{v} + u_i \int w_k f d^3\mathbf{v} + \int w_k w_i f d^3\mathbf{v}] \\
&= \frac{\partial}{\partial x_k}[n u_k u_i + 0 + 0 + \frac{1}{m}\mathrm{P}_{ki}] \\
&= \nabla \cdot (n\mathbf{u}\mathbf{u}) + \frac{1}{m}\nabla \cdot \mathbf{P}
\end{aligned} \tag{4}$$

where we have made the decomposition:

$$\mathbf{v} = \mathbf{u}(\mathbf{x}, t) + \mathbf{w}(\mathbf{v}, \mathbf{x}, t) \tag{5}$$

and the pressure tensor is defined as

$$\mathbf{P} = m \int \mathbf{w}\mathbf{w} f d^3 v \tag{6}$$

2.3

$$
\begin{aligned}
\int \mathbf{E} \cdot \frac{\partial f}{\partial \mathbf{v}} \mathbf{v} d^3\mathbf{v} &= \int E_k \frac{\partial f}{\partial v_k} v_i d^3\mathbf{v} \\
&= -\int f \frac{\partial (E_k v_i)}{\partial v_k} d^3\mathbf{v} \\
&= -\int f E_k \delta_{ik} d^3\mathbf{v} \\
&= -E_i \int f d^3\mathbf{v} \\
&= -n\mathbf{E}
\end{aligned} \tag{7}
$$

2.4

Similarly

$$
\begin{aligned}
\int (\mathbf{v} \times \mathbf{B}) \cdot \frac{\partial f}{\partial \mathbf{v}} \mathbf{v} d^3\mathbf{v} &= -\int f (\mathbf{v} \times \mathbf{B})_k \delta_{ik} d^3\mathbf{v} \\
&= -n(\mathbf{u} \times \mathbf{B})
\end{aligned} \tag{8}
$$

In summary, the 1st order momentum equation, or, the **momentum-conservation equation**, is

$$
\frac{\partial}{\partial t}(nm\mathbf{u}) + \nabla \cdot (nm\mathbf{u}\mathbf{u}) = -\nabla \cdot \mathbf{P} + nq(\mathbf{E} + \mathbf{u} \times \mathbf{B}) + \mathbf{R} \tag{9}
$$

where $\mathbf{R} = m \int (\frac{\partial f}{\partial t})_c \mathbf{v} d^3\mathbf{v}$ is the collision term.

3 Equation for the pressure and energy

Again we need to make the decomposition Eq (5) in the derivation such that

$$
v_k v_i v_j = u_k u_i u_j + (u_k u_i w_j + u_k w_i u_j + w_k u_i u_j) + (u_k w_i w_j + w_k u_i w_j + w_k w_i u_j) + w_k w_i w_j \tag{10}
$$

3.1

$$
\begin{aligned}
\frac{\partial}{\partial t} \int d^3\mathbf{v} f \mathbf{v}\mathbf{v} &= \frac{\partial}{\partial t}(n u_i u_j + \frac{1}{m} \mathrm{P}_{ij}) \\
&= \frac{1}{m} \frac{\partial}{\partial t}(nm\mathbf{u}\mathbf{u} + \mathbf{P})
\end{aligned} \tag{11}
$$

3.2

$$
\begin{aligned}
\int d^3\mathbf{v} v_k \frac{\partial f}{\partial x_k} v_i v_j &= \frac{\partial}{\partial x_k}(\int d^3\mathbf{v} f v_k v_i v_j) \\
&= \frac{\partial}{\partial x_k} \int d^3\mathbf{v} f [u_k u_i u_j + (u_k u_i w_j + u_k w_i u_j + w_k u_i u_j) + (u_k w_i w_j + w_k u_i w_j + w_k w_i u_j) + w_k w_i w_j] \\
&= \frac{\partial}{\partial x_k}(n u_k u_i u_j) + \frac{1}{m} \frac{\partial}{\partial x_k}[u_k \mathrm{P}_{ij} + u_i \mathrm{P}_{kj} + u_j \mathrm{P}_{ki}] + \frac{1}{m} \frac{\partial}{\partial x_k}(m \int d^3\mathbf{v} f w_k w_i w_j)
\end{aligned} \tag{12}
$$

3.3

$$
\begin{aligned}
\int \mathbf{E} \cdot \frac{\partial f}{\partial \mathbf{v}} \mathbf{v}\mathbf{v} d^3\mathbf{v} &= -\int f E_k \frac{\partial}{\partial v_k}(v_i v_j) d^3\mathbf{v} \\
&= -\int f E_k (\delta_{ik} v_j + v_i \delta_{jk}) d^3\mathbf{v} \\
&= -\int f (E_i v_j + v_i E_j) d^3\mathbf{v} \\
&= -n(\mathbf{E}\mathbf{u} + \mathbf{u}\mathbf{E})
\end{aligned} \tag{13}
$$

3.4

$$\int (\mathbf{v}\times\mathbf{B})\cdot\frac{\partial f}{\partial \mathbf{v}}\mathbf{v}\mathbf{v}d^3\mathbf{v} = -\int d^3\mathbf{v}[f(\mathbf{v}\times\mathbf{B})_i v_j + f v_i(\mathbf{v}\times\mathbf{B})_j] \tag{14}$$

Then we can do the decomposition:

$$(\mathbf{v}\times\mathbf{B})_i v_j = (\mathbf{u}\times\mathbf{B})_i u_j + (\mathbf{w}\times\mathbf{B})_i w_j + (\mathbf{u}\times\mathbf{B})_i w_j + (\mathbf{w}\times\mathbf{B})_i u_j \tag{15}$$

where the last two terms on L.H.S. vanish after the integral. Similar decomposition is applied to the term $v_i(\mathbf{v}\times\mathbf{B})_j$. It is then easy to verify

$$\int (\mathbf{v}\times\mathbf{B})\cdot\frac{\partial f}{\partial \mathbf{v}}\mathbf{v}\mathbf{v}d^3\mathbf{v} = -n[(\mathbf{u}\times\mathbf{B})\mathbf{u} + \mathbf{u}(\mathbf{u}\times\mathbf{B})] - \int f[(\mathbf{w}\times\mathbf{B})_i w_j + w_i(\mathbf{w}\times\mathbf{B})_j]d^3\mathbf{v} \tag{16}$$

3.5

The collision term $\int(\frac{\partial f}{\partial t})_c\mathbf{v}\mathbf{v}d^3\mathbf{v}$ needs further treatment. For now we assume it is 0.

Sum up all the terms, we get the final form for the **pressure tensor equation**:

$$\begin{aligned}&\frac{\partial}{\partial t}(nm\mathbf{u}\mathbf{u} + \mathbf{P}) + \nabla\cdot(nm\mathbf{u}\mathbf{u}\mathbf{u}) + \nabla\cdot(\mathbf{u}\mathbf{P}) + \frac{\partial}{\partial x_k}(u_i\mathrm{P}_{kj} + u_j\mathrm{P}_{ki}) + \frac{\partial}{\partial x_k}(m\int f w_k w_i w_j d^3\mathbf{v}) \\ &= qn(\mathbf{E}\mathbf{u} + \mathbf{u}\mathbf{E}) + qn[(\mathbf{u}\times\mathbf{B})\mathbf{u} + \mathbf{u}(\mathbf{u}\times\mathbf{B})] + q\int f[(\mathbf{w}\times\mathbf{B})_i w_j + w_i(\mathbf{w}\times\mathbf{B})_j]d^3\mathbf{v}\end{aligned} \tag{17}$$

However, Eq (17) is a tensor equation which is too complex to use. In practice, we often use the scalar energy density instead of the pressure tensor for simplicity. To derive the energy equation, we take the trace of Eq (17). One can verify the following relations:

$$\mathrm{Tr}[\frac{\partial}{\partial t}(nm\mathbf{u}\mathbf{u})] = \frac{\partial}{\partial t}(nmu^2) \tag{18a}$$

$$\mathrm{Tr}[\frac{\partial}{\partial t}\mathbf{P}] = \frac{\partial}{\partial t}[\mathrm{Tr}(\mathbf{P})] \tag{18b}$$

$$\mathrm{Tr}[\nabla\cdot(nm\mathbf{u}\mathbf{u}\mathbf{u})] = \frac{\partial}{\partial x_k}(nmu_k u^2) = \nabla\cdot(nmu^2\mathbf{u}) \tag{18c}$$

$$\mathrm{Tr}[\nabla\cdot(\mathbf{u}\mathbf{P})] = \frac{\partial}{\partial x_k}[u_k\,\mathrm{Tr}(\mathbf{P})] = \nabla\cdot[\mathbf{u}\,\mathrm{Tr}(\mathbf{P})] \tag{18d}$$

$$\mathrm{Tr}[\frac{\partial}{\partial x_k}(u_i\mathrm{P}_{kj} + u_j\mathrm{P}_{ki})] = \frac{\partial}{\partial x_k}(u_i\mathrm{P}_{ki} + u_i\mathrm{P}_{ki}) = 2\nabla\cdot(\mathbf{P}\cdot\mathbf{u}) \tag{18e}$$

$$\mathrm{Tr}[\frac{\partial}{\partial x_k}(m\int f w_k w_i w_j d^3\mathbf{v})] = \frac{\partial}{\partial x_k}(m\int f w_k w^2 d^3\mathbf{v}) = \nabla\cdot(\int f m w^2\mathbf{w}d^3\mathbf{v}) \tag{18f}$$

$$\mathrm{Tr}[qn(\mathbf{E}\mathbf{u} + \mathbf{u}\mathbf{E})] = 2qn(\mathbf{E}\cdot\mathbf{u}) \tag{18g}$$

$$\mathrm{Tr}[qn[(\mathbf{u}\times\mathbf{B})\mathbf{u} + \mathbf{u}(\mathbf{u}\times\mathbf{B})]] = 2qn[\mathbf{u}\cdot(\mathbf{u}\times\mathbf{B})] = 0 \tag{18h}$$

$$\mathrm{Tr}[q\int f[(\mathbf{w}\times\mathbf{B})_i w_j + w_i(\mathbf{w}\times\mathbf{B})_j]d^3\mathbf{v}] = 2q\int f\mathbf{w}\cdot(\mathbf{w}\times\mathbf{B})d^3\mathbf{v} = 0 \tag{18i}$$

Sum up all the terms and divide the equation by 2, we get the final form of the **energy equation**:

$$\frac{\partial}{\partial t}[\frac{1}{2}nmu^2 + \frac{1}{2}\mathrm{Tr}(\mathbf{P})] + \nabla\cdot[\frac{1}{2}nmu^2\mathbf{u} + \frac{1}{2}\mathrm{Tr}(\mathbf{P})\mathbf{u}] + \nabla\cdot(\mathbf{P}\cdot\mathbf{u}) + \nabla\cdot[\int f\frac{1}{2}mw^2\mathbf{w}d^3\mathbf{v}] = qn(\mathbf{E}\cdot\mathbf{u}) \tag{19}$$

and we can define the *heat flux* by

$$\mathbf{Q} = \int f\frac{1}{2}mw^2\mathbf{w}d^3\mathbf{v} \tag{20}$$

The collision term, which is usually called *heat conduction*, is neglected in the absence of collisions.