Derivation of Klimontovich equation

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For one species (mass $m$ and charge $q$), the status of the system can be written as:

$$G(x, v, t) = \sum_{i=1}^{N} \delta(x - x_i(t)) \delta(v - v_i(t))$$  \hspace{1cm} (1)

If we take time derivative of $G$, we get:

$$\frac{\partial G}{\partial t} = \sum_{i=1}^{N} \left[ \delta(v - v_i(t)) \frac{\partial \delta(x - x_i(t))}{\partial x_i} \frac{dx_i}{dt} + \delta(x - x_i(t)) \frac{\partial \delta(v - v_i(t))}{\partial v_i} \frac{dv_i}{dt} \right]$$

$$= -\left[ \delta(v - v_i(t)) v_i \cdot \frac{\partial \delta(x - x_i(t))}{\partial x} + \delta(x - x_i(t)) \frac{F(x_i, v_i, t)}{m} \cdot \frac{\partial \delta(v - v_i(t))}{\partial v} \right]$$  \hspace{1cm} (2)

where we have used the relation:

$$\frac{\partial \delta(a - b)}{\partial a} = -\frac{\partial \delta(a - b)}{\partial b}$$

Then we use another relation:

$$f(a)\delta(a - b) = f(b)\delta(a - b)$$

so that

$$\delta(v - v_i(t)) v_i \cdot \frac{\partial \delta(x - x_i(t))}{\partial x} = \delta(v - v_i(t)) v \cdot \frac{\partial \delta(x - x_i(t))}{\partial x}$$

$$= v \cdot \nabla_x \left[ \delta(v - v_i(t)) \delta(x - x_i(t)) \right]$$

and

$$\delta(x - x_i(t)) \frac{F(x_i, v_i, t)}{m} \cdot \frac{\partial \delta(v - v_i(t))}{\partial v} = \delta(x - x_i(t)) \frac{F(x, v, t)}{m} \cdot \frac{\partial \delta(v - v_i(t))}{\partial v}$$

$$= \delta(x - x_i(t)) \nabla_v \left[ \frac{F(x, v, t)}{m} \delta(v - v_i(t)) \right]$$

$$= \delta(x - x_i(t)) \nabla_v \left[ \frac{F(x, v, t)}{m} \delta(v - v_i(t)) \right]$$

Note that

$$F(x, v, t) = E(x, t) + qv \times B(x, t)$$

so

$$\nabla_v \cdot F(x, v, t) = 0$$

which means

$$\delta(x - x_i(t)) \frac{F(x_i, v_i, t)}{m} \cdot \frac{\partial \delta(v - v_i(t))}{\partial v} = \delta(x - x_i(t)) \frac{F(x, v, t)}{m} \cdot \nabla_v \delta(v - v_i(t))$$

$$= \frac{F(x, v, t)}{m} \cdot \nabla_v G$$

Finally, we get the equation

$$\frac{\partial G}{\partial t} + v \cdot \nabla_x G + \frac{F(x, v, t)}{m} \cdot \nabla_v G = 0$$  \hspace{1cm} (3)