Derivation of Klimontovich equation

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For one species (mass m and charge q), the status of the system can be written as:

$$G(\mathbf{x}, \mathbf{v}, t) = \sum_{i=1}^{N} \delta(\mathbf{x} - \mathbf{x_i}(t))\delta(\mathbf{v} - \mathbf{v_i}(t)) \tag{1}$$

If we take time derivative of G, we get:

$$\begin{aligned} \frac{\partial G}{\partial t} &= \sum_{i=1}^{N} \big[\delta(\mathbf{v} - \mathbf{v_i}(t)) \frac{\partial \delta(\mathbf{x} - \mathbf{x_i}(t))}{\partial \mathbf{x_i}} \cdot \frac{d\mathbf{x_i}}{dt} + \delta(\mathbf{x} - \mathbf{x_i}(t)) \frac{\partial \delta(\mathbf{v} - \mathbf{v_i}(t))}{\partial \mathbf{v_i}} \cdot \frac{d\mathbf{v_i}}{dt}\big] \\ &= -\big[\delta(\mathbf{v} - \mathbf{v_i}(t))\mathbf{v_i} \cdot \frac{\partial \delta(\mathbf{x} - \mathbf{x_i}(t))}{\partial \mathbf{x}} + \delta(\mathbf{x} - \mathbf{x_i}(t)) \frac{\mathbf{F}(\mathbf{x_i}, \mathbf{v_i}, t)}{m} \cdot \frac{\partial \delta(\mathbf{v} - \mathbf{v_i}(t))}{\partial \mathbf{v}}\big] \end{aligned} \tag{2}$$

where we have used the relation:

$$\frac{\partial \delta(a-b)}{\partial a} = -\frac{\partial \delta(a-b)}{\partial b}$$

Then we use another relation:

$$f(a)\delta(a-b) = f(b)\delta(a-b)$$

so that

$$\begin{aligned} \delta(\mathbf{v} - \mathbf{v_i}(t))\mathbf{v_i} \cdot \frac{\partial \delta(\mathbf{x} - \mathbf{x_i}(t))}{\partial \mathbf{x}} &= \delta(\mathbf{v} - \mathbf{v_i}(t))\mathbf{v} \cdot \frac{\partial \delta(\mathbf{x} - \mathbf{x_i}(t))}{\partial \mathbf{x}} \\ &= \mathbf{v} \cdot \nabla_{\mathbf{x}} \big[\delta(\mathbf{v} - \mathbf{v_i}(t))\delta(\mathbf{x} - \mathbf{x_i}(t))\big] \\ &= \mathbf{v} \cdot \nabla_{\mathbf{x}} G \end{aligned}$$

and

$$\begin{aligned} \delta(\mathbf{x} - \mathbf{x_i}(t)) \frac{\mathbf{F}(\mathbf{x_i}, \mathbf{v_i}, t)}{m} \cdot \frac{\partial \delta(\mathbf{v} - \mathbf{v_i}(t))}{\partial \mathbf{v}} &= \delta(\mathbf{x} - \mathbf{x_i}(t)) \frac{\mathbf{F}(\mathbf{x}, \mathbf{v_i}, t)}{m} \cdot \frac{\partial \delta(\mathbf{v} - \mathbf{v_i}(t))}{\partial \mathbf{v}} \\ &= \delta(\mathbf{x} - \mathbf{x_i}(t)) \nabla_{\mathbf{v}} \cdot [\frac{\mathbf{F}(\mathbf{x}, \mathbf{v_i}, t)}{m} \delta(\mathbf{v} - \mathbf{v_i}(t))] \\ &= \delta(\mathbf{x} - \mathbf{x_i}(t)) \nabla_{\mathbf{v}} \cdot [\frac{\mathbf{F}(\mathbf{x}, \mathbf{v}, t)}{m} \delta(\mathbf{v} - \mathbf{v_i}(t))] \end{aligned}$$

Note that

$$\mathbf{F}(\mathbf{x}, \mathbf{v}, t) = \mathbf{E}(\mathbf{x}, t) + q\mathbf{v} \times \mathbf{B}(\mathbf{x}, t)$$

so

$$\nabla_{\mathbf{v}} \cdot \mathbf{F}(\mathbf{x}, \mathbf{v}, t) = 0$$

which means

$$\begin{aligned} \delta(\mathbf{x} - \mathbf{x_i}(t)) \frac{\mathbf{F}(\mathbf{x_i}, \mathbf{v_i}, t)}{m} \cdot \frac{\partial \delta(\mathbf{v} - \mathbf{v_i}(t))}{\partial \mathbf{v}} &= \delta(\mathbf{x} - \mathbf{x_i}(t)) \frac{\mathbf{F}(\mathbf{x}, \mathbf{v}, t)}{m} \cdot \nabla_{\mathbf{v}} \delta(\mathbf{v} - \mathbf{v_i}(t)) \\ &= \frac{\mathbf{F}(\mathbf{x}, \mathbf{v}, t)}{m} \cdot \nabla_{\mathbf{v}} G \end{aligned}$$

Finally, we get the equation

$$\frac{\partial G}{\partial t} + \mathbf{v} \cdot \nabla_{\mathbf{x}} G + \frac{\mathbf{F}(\mathbf{x}, \mathbf{v}, t)}{m} \cdot \nabla_{\mathbf{v}} G = 0 \tag{3}$$