Hall MHD and the whistler/ion-cyclotron waves

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1 Generalized Ohm's Law and the normalization

The generalized Ohm's Law can be written as:

$$\mathbf{E} + \mathbf{u} \times \mathbf{B} = \eta \mathbf{j} + \frac{1}{ne}\mathbf{j} \times \mathbf{B} - \frac{1}{ne}\nabla \cdot \mathbf{P_e} + \frac{m_e}{ne^2}\left[\frac{\partial \mathbf{j}}{\partial t} + \nabla \cdot (\mathbf{uj} + \mathbf{ju} - \frac{\mathbf{jj}}{ne})\right] \tag{1}$$

If the electron temperature is very low and the frequency of the current fluctuation is small, we can keep only the resistive term and the Hall term in the R.H.S.:

$$\mathbf{E} + \mathbf{u} \times \mathbf{B} = \eta \mathbf{j} + \frac{1}{ne}\mathbf{j} \times \mathbf{B} \tag{2}$$

Plug it in the Faraday equation, we get

$$\frac{\partial \mathbf{B}}{\partial t} = \nabla \times (\mathbf{u} \times \mathbf{B}) - \nabla \times (\frac{1}{ne}\mathbf{j} \times \mathbf{B}) - \nabla \times (\eta \mathbf{j}) \tag{3}$$

Choose the characteristic quantities: magnetic field $\bar{B}$, number density $\bar{n}$, length L for normalization. The characteristic speed, time and current are: the Alfvén speed $V_A = \bar{B}/\sqrt{\mu_0 \bar{n} m_i}$, the Alfvén time $\tau_A = L/V_A$ and the current $\bar{j} = \bar{B}/\mu_0 L$. Then Eq (3) can be written as the dimensionless form:

$$\frac{\partial \mathbf{B}}{\partial t} = \nabla \times (\mathbf{u} \times \mathbf{B}) - \frac{d_i}{L}\nabla \times (\frac{\mathbf{j} \times \mathbf{B}}{n}) - \frac{1}{S}\nabla \times \mathbf{j} \tag{4}$$

where the normalized current is

$$\mathbf{j} = \nabla \times \mathbf{B} \tag{5}$$

d_i is the ion inertial length:

$$d_i = \frac{V_A}{\omega_{ci}} = \frac{c}{\omega_{pi}} = \sqrt{\frac{m_i}{\mu_0 \bar{n} e^2}} \tag{6}$$

and S is the Lundquist number:

$$S = \frac{L V_A \mu_0}{\eta} \tag{7}$$

Similarly, if we choose the same normalization and choose $\bar{n} m_i V_A^2$ to normalize the pressure, we can write the momentum equation in dimensionless form:

$$n(\frac{\partial \mathbf{u}}{\partial t} + \mathbf{u} \cdot \nabla \mathbf{u}) = -\nabla p + \mathbf{j} \times \mathbf{B} \tag{8}$$

2 Dispersion relation for whistler/ion-cyclotron waves

In ideal MHD, there are only 3 types of waves: Alfvén wave, fast magnetosonic wave and slow magnetosonic wave. However, if we include the Hall term, we will get two more waves: whistler wave and the ion cyclotron wave.

We start from the two normalized equations:

$$n(\frac{\partial \mathbf{u}}{\partial t} + \mathbf{u} \cdot \nabla \mathbf{u}) = -\nabla p + \mathbf{j} \times \mathbf{B} \tag{9a}$$

$$\frac{\partial \mathbf{B}}{\partial t} = \nabla \times (\mathbf{u} \times \mathbf{B}) - \frac{d_i}{L}\nabla \times (\frac{\mathbf{j} \times \mathbf{B}}{n}) - \frac{1}{S}\nabla \times \mathbf{j} \tag{9b}$$

We do not consider the resistive term and we assume uniform background fields $\mathbf{B_0} = 1$, $n_0 = 1$ and $\mathbf{u_0} = 0$. We also assume that the perturbation is incompressible so $n_1 = 0$ and $\mathbf{k} \cdot \mathbf{u_1} = 0$ where $\mathbf{k}$ is the wave vector. For simplicity write d_i/L as d_i so that d_i is also normalized to L. Take curl of the momentum equation and then it is easy to write the linearized equation

$$\frac{\partial}{\partial t}(\nabla \times \mathbf{u_1}) = \nabla \times (\mathbf{j_1} \times \mathbf{B_0}) \tag{10a}$$

$$\frac{\partial \mathbf{b_1}}{\partial t} = \nabla \times (\mathbf{u_1} \times \mathbf{B_0}) - d_i \nabla \times (\mathbf{j_1} \times \mathbf{B_0}) \tag{10b}$$

Fourier transform in space and time. Because $\mathbf{B_0}$ is constant, we can directly replace $\partial/\partial t$ and ∇ to $-i\omega$ and $\mathbf{k}$ in Eq (10). We then get

$$\omega \mathbf{k} \times \mathbf{u_1} = -(\mathbf{k} \cdot \mathbf{B_0})(\mathbf{k} \times \mathbf{b_1}) \tag{11a}$$

$$\mathbf{b_1} = -\frac{\mathbf{k} \cdot \mathbf{B_0}}{\omega}\mathbf{u_1} - i d_i (\mathbf{k} \times \mathbf{u_1}) \tag{11b}$$

Use the 2nd equation to eliminate $\mathbf{b_1}$ in the 1st equation, we get

$$\left[\omega - \frac{(\mathbf{k} \cdot \mathbf{B_0})^2}{\omega}\right](\mathbf{k} \times \mathbf{u_1}) + i d_i k^2 (\mathbf{k} \cdot \mathbf{B_0})\mathbf{u_1} = 0 \tag{12}$$

$\mathbf{k}\times$Eq (12), we get

$$-k^2\left[\omega - \frac{(\mathbf{k} \cdot \mathbf{B_0})^2}{\omega}\right]\mathbf{u_1} + i d_i k^2 (\mathbf{k} \cdot \mathbf{B_0})(\mathbf{k} \times \mathbf{u_1}) = 0 \tag{13}$$

Combine Eq (12) and (13) and let the determinant of the coefficient matrix to be 0, we get the equation for the dispersion relation:

$$\left[\omega - \frac{(\mathbf{k} \cdot \mathbf{B_0})^2}{\omega}\right]^2 = [k d_i (\mathbf{k} \cdot \mathbf{B_0})]^2 \tag{14}$$

whose solution is:

$$\omega = \frac{\pm k d_i \pm \sqrt{(k d_i)^2 + 4}}{2}(\mathbf{k} \cdot \mathbf{B_0}) \tag{15}$$

Note that the "++" and the "−−" solutions are the same except for the propagation direction of the wave (and similarly "+−" and "−+" are the same). In conclusion, the dispersion relation is

$$\omega_\pm = \frac{\sqrt{(k d_i)^2 + 4} \pm k d_i}{2}(\mathbf{k} \cdot \mathbf{B_0}) \tag{16}$$

If we transform from the normalized quantities to the real quantities, we get

$$\omega_\pm = \frac{\sqrt{(k d_i)^2 + 4} \pm k d_i}{2}(\mathbf{k} \cdot \mathbf{V_A}) \tag{17}$$

2.1 Discussion

One immediate result from Eq (17) is that if $k d_i = 0$, we get

$$\omega = \mathbf{k} \cdot \mathbf{V_A} \tag{18}$$

i.e., the two waves degenerate to Alfvén wave. Then, let's look at the polarization of the two branches of the dispersion relation. From Eq (12), we see that the polarization is determined by the polarity of

$$\alpha = \omega^2 - (\mathbf{k} \cdot \mathbf{B_0})^2 \tag{19}$$

It is easy to show from Eq (16) that $\alpha(\omega_+) > 0$ and $\alpha(\omega_-) < 0$. From Eq (12), we see that, if $\alpha > 0$ (ω_+), $\mathbf{k} \times \mathbf{u_1} \propto \exp(-i\omega t - \pi/2)$ is $\pi/2$ faster than $\mathbf{u_1}$, which means the wave is right-hand polarized (electron cyclotron wave, or whistler wave). Vice versa, if ω_-, the wave is left-hand polarized (ion cyclotron wave).

If we take the limit that $k \to \infty$, we get (assume $\mathbf{k}$ is along $\mathbf{V_A}$)

$$\begin{aligned} \omega_+ &= k^2 d_i V_A \\ &= k^2 d_i^2 \omega_{ci} \\ &= k^2 d_e^2 \omega_{ce} \end{aligned} \tag{20a}$$

$$\begin{aligned} \omega_- &= \frac{V_A}{d_i} \\ &= \omega_{ci} \end{aligned} \tag{20b}$$

Apparently Eq (20a) is the dispersion relation for whistler wave and Eq (20b) is the limit of the dispersion relation of ion-cyclotron wave. The plot of the dispersion relation is shown in Fig (1).

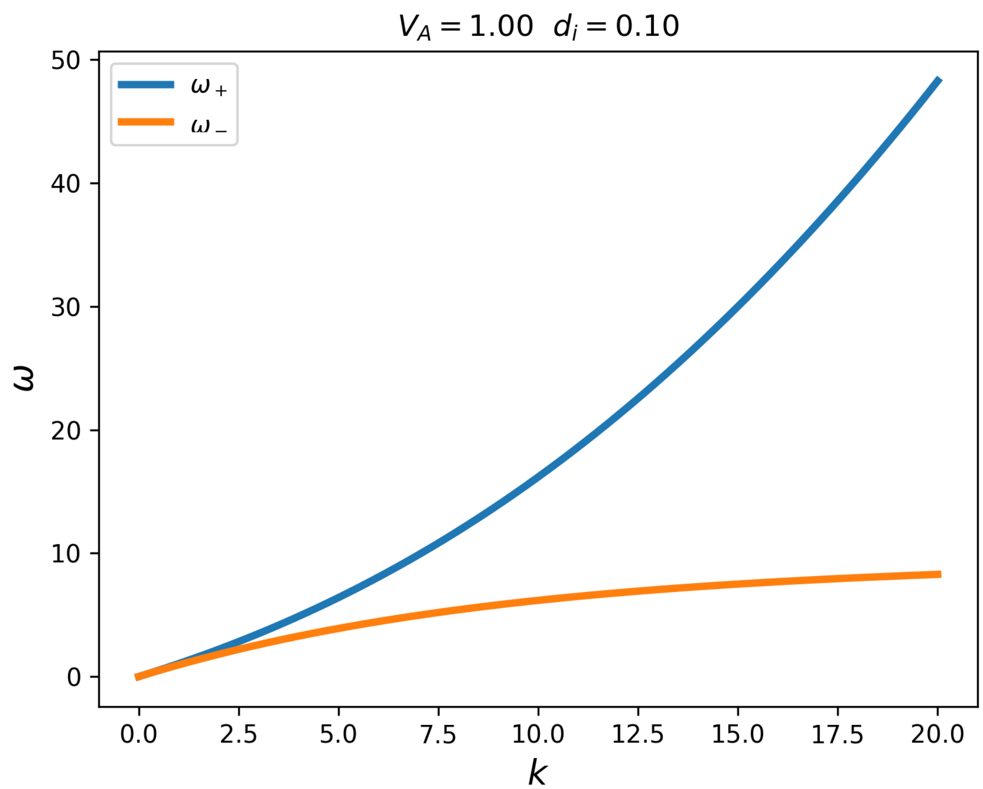

Figure 1: Dispersion relation Eq (17). $V_A = 1.0$ and $d_i = 0.1$